

# Singularity: A MAPLE library for local zeros of scalar smooth maps

Majid Gazor\* and Mahsa Kazemi

*Department of Mathematical Sciences, Isfahan University of Technology,  
Isfahan 84156-83111, Iran*

## Abstract

The local zero structure of a smooth map may *qualitatively* change, when the map is subjected to small perturbations. The changes may include births and/or deaths of zeros. The *qualitative properties* are defined as the invariances of an appropriate *equivalence relation*. The occurrence of a *qualitative change* in the zero structures is called a *bifurcation* and the map is named a *singularity*. The local bifurcation analysis of singularities has been extensively studied in *singularity theory* and many powerful algebraic tools have been developed for their study. However, there does not exist any available symbolic computer-library for this purpose. We suitably generalize some powerful tools from algebraic geometry for correct implementation of the results from singularity theory. We provide some required criteria along with rigorous proofs for efficient and cognitive computer-implementation. We have accordingly developed a MAPLE end-user friendly library, named “**Singularity**”, for an efficient and complete local bifurcation analysis of real zeros of scalar smooth maps. We have further constructed a built-in help for **Singularity** and provided a comprehensive user-guide. Recent progresses in MAPLE have been used so that **Singularity** generates the list of inequivalent persistent bifurcation diagrams for parametric singularities. The approach is independent of the *singularity’s codimension* (*a measure of its complexity*). The main features of **Singularity** are briefly illustrated along with a few examples. Our MAPLE library is not only useful for research goals and engineering applications involving singularities but also for pedagogical purposes.

**Keywords:** Singularity and bifurcation theory; Ideal membership problem; Standard and Gröbner bases.

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\* Corresponding author. Phone: (98-31) 33913634; Fax: (98-31) 33912602; Email: mgazor@cc.iut.ac.ir.

Many real life problems may result in the analysis of local zeros (around a zero solution, named a *base point*) of a smooth map

$$f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n, \quad f(x, \alpha) = 0, \quad \text{for } x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m. \quad (0.1)$$

We refer to  $x$  by state variables,  $n$  by state dimension and  $\alpha$  by parameters. Note that locating singular base points of a smooth map is related to finding roots of nonlinear systems and is not the purpose of our work here; see [17, 30, 31]. Hence, we may assume that the base point is the origin and  $f(0, 0) = 0$ . Equation (0.1) may demonstrate a surprising change on the solution set when the parameters vary. This occurs when the Jacobian matrix of  $f$  does not have a full rank. In this case we say that  $f$  is *singular*.

Equation (0.1) may appear by direct mathematical modeling of a singular engineering problem or indirect, through reduction methods such as *Liapunov-Schmidt reduction*; e.g., see [30], [31, Pages 156–162] and [29, Chapter VII]. For example, Equation (0.1) appears in the study of equilibria and limit cycles of dynamical systems or steady-state solutions of partial differential equations. In fact, the theory described here is known as a “natural framework” for *equilibrium bifurcation theory*; see [30]. Using Liapunov-Schmidt reduction, we can reduce the state dimension so that the Jacobian matrix at the origin is the zero matrix. Thus in this paper, we assume that

$$n = 1 \quad \text{and} \quad \frac{\partial f}{\partial x}(0, 0) = 0. \quad (0.2)$$

We will deal with the case of multi-state dimensional problems in a future project. When  $f$  is a singular map and the parameters  $\alpha$  vary, the number of solutions for Equation (0.1) may change and any of such changes is called a *bifurcation*. The equation  $f(x, \alpha) = 0$  is called a *bifurcation problem* and the set

$$\{(x, \alpha) : f(x, \alpha) = 0\} \quad (0.3)$$

is called a *bifurcation diagram*.

Singularities (including those different from here) and the theories to deal with them have appeared in several mathematical disciplines with the same or similar keywords. These theories have a long history dating back to the original works of Hassler Whitney, and then John Mather, Rene Thom and V.I. Arnold in late 1960s. Depending on the context, space and equivalence relation, they are faced with essentially different tools, algebraic and geometric structures, applications and/or objectives; e.g., see [2, 7, 15, 29, 32, 44, 45]. For an instance a *geometric singularity* refers to a variety whose tangent space is not regularly defined; e.g., a self intersecting curve or a cusp; see [15, 32]. This leads to a theory of its own and it is named *singularity theory*. Arnold in [1, 2] uses  $\mu$ - and *right*-equivalences for his normal form classification of smooth functions. *Strategy equivalence* is used for the analysis of *strategy functions* in [45]. *Right-left* equivalence is applied in [7] to treat bifurcations of Hamiltonian systems. The other instance is *catastrophe theory* [44] and usually uses almost identical keywords as ours. Catastrophe theory addresses basically different qualitative properties

using *right equivalence* than the qualitative properties we pursue here using *contact equivalence*. All of these mentioned subject areas follow different objectives and goals than what ours do. In this paper we only refer to the methodologies in dealing with the local bifurcation analysis of real zeros of smooth maps (0.1) as singularity theory; see [3, 4, 18–20, 29, 31, 38, 39, 41] for our main references of this subject.

Many powerful algebraic tools have been developed for local bifurcation analysis of zeros in Equation (0.1). Armbruster [3] proposed a cognitive use of Gröbner basis and encouraged a systematic implementations of the existing results in a computer. Yet to the best of our knowledge, there does not exist any end-user symbolic library available (neither free nor paid) to everyone for the *local bifurcation analysis of zeros of smooth maps*. This is a long overdue contribution despite its importance and wide applications. In the last two decades, there has been a considerable progress in development of computer algebra systems so that an efficient *symbolic* implementation of the results in singularity theory is now feasible; see [30, 31] for numeric approaches.

Singularity theory uses two fundamental notions of *intermediate order terms* and *high order terms* for *normal form computation*. Recall that theory of normal forms here is to simplify singular maps and qualitative study of their zeros. Intermediate order terms have a complicated structure and do not seem to hold a convenient algebraic structure. Therefore, Gröbner basis type of tools (i.e., its local versions) alone are not sufficient to simplify removable intermediate order terms in a singular germ. Furthermore as far as our information is concerned, there does not yet exist a complete algebraic characterization for *high order terms* in many cases of multi-state variables. This does not mean that high and intermediate order terms are computationally inaccessible but it does mean crude uses of Gröbner basis type of tools fail to address *normal form computations*. (Note that the basic and principal ideas in singularity theory start with normal form computation.) Indeed, the approach needs to be refined along with nonlinear normalization techniques. Our approach efficiently simplifies removable intermediate order terms and is adoptable for simplification of high order terms even for the generalization of our results to multi-state variable cases.

Our main contributions here are some natural generalizations of known tools from algebraic geometry for correct symbolic implementation and providing certain criteria along with rigorous proofs for alternative and efficient symbolic implementation, an efficient symbolic implementation of the results of singularity theory (one dimensional state variable) in MAPLE, developing an end-user friendly MAPLE library named **Singularity**, writing a user guide [21] and constructing a comprehensive built-in MAPLE help for **Singularity**.

A contribution here is to generalize some existing techniques and concepts from algebraic geometry to the context of (locally) smooth maps (germs) for a correct implementation. Due to the infinite nature of Taylor series of smooth maps, the computations are performed modulo a given degree. We provide a sufficient condition for a given degree whose truncation does not lead to error. The default work of **Singularity** tests the condition and do the computations modulo an

optimal degree. However, this approach adds a computational cost. Further, smooth maps involve flat functions (functions with zero Taylor series) and this may cause unnecessarily complicated formulas. Thereby, it is fundamentally helpful to use a ring smaller than the ring of smooth maps when it is feasible. Unlike the ring of formal power series, the associated computations in the ring of polynomials or fractional maps are exact and no truncation is required. We provide conditions along with rigorous proofs for the possible efficient implementations using ideals generated in either the rings of polynomials, fractional maps or formal power series. **Singularity** is adopted accordingly. We accordingly provide guidelines for efficient symbolic implementation of the existing results from singularity theory; see [31, Chapters 6–7], [41, Sections 6.2 and 6.3], [29, Chapters 1–4] and [3, 4, 19, 39]. This simply illustrates an interesting and realistic application of computational algebraic geometry to the equilibrium bifurcation analysis of real world problems.

MAPLE is one of the two major (along with MATHEMATICA) symbolic computer algebra systems. Besides, MAPLE is user-friendly and is widely used for Mathematics and engineering research as well as educational purposes. Thus, MAPLE is a good choice for end-user libraries in general, and in particular for **Singularity** due to various applications of singularity theory in different engineering disciplines. Further, many useful and advanced techniques from computational algebraic geometry such as Gröbner basis, elimination ideals for polynomial ring, regular chains and triangular decomposition method have already been developed and are now available in built-in packages of MAPLE 18. So far, the most efficient Gröbner basis computation is due to MAPLE  $F_4$  algorithm implemented by Jean-Charles Faugère (see [33]) while regular chains and triangular decomposition method are only available in MAPLE. **Singularity** has benefited from an efficient use of these capacities of MAPLE. Besides, we have already developed some MAPLE programs for (parametric and orbital) normal form computation of *singular differential equations* [22–27] and their integration with **Singularity** shall lead to a toolbox for local bifurcation control and analysis of singularities. **Singularity** has been tested by all scalar examples (nonsymmetric and without modal parameters) and classifications given in [29, 31, 38, 39] and a few (differences, error or not already reported data due to computational burden) are verified in our favor. We use identical notations and terminologies from [15, 29, 31, 32, 38, 39] as far as it is feasible.

**Singularity** computes a variety of algebraic structures associated with singular scalar maps including *tangent* and *restricted tangent spaces*, *high order term ideals*, and the *intrinsic ideals* associated with ideals of both *finite* and *infinite codimension*. Our MAPLE library derives *low and intermediate order terms*, *normal forms*, *universal unfolding*, and *transition sets*. **Singularity** efficiently simplifies *intermediate* and *high order terms*. It, further, generates *persistent bifurcation diagrams (plot or animation)*, and estimates the *transformations* transforming *contact-equivalent scalar maps to each other*. Finally, **Singularity** solves *the recognition problem for normal forms and universal unfoldings*. An interesting capability of **Singularity** is the classification of persistent bifurcation diagrams by generating an automatic list. This latter capability is in fact enabled by

using a powerful built-in MAPLE 18 package called **RegularChains** [12].

In order to develop **Singularity**, we have discussed some modified concepts (suitable in our context) from computational algebraic geometry including *division remainders*, *standard bases*, *elimination ideals*, *ideal membership problem* and *colon ideals* for the local rings of fractional (germ) maps, formal power series and smooth maps. These are accordingly implemented in **Singularity**.

The rest of this paper is organized as follows. Singularity theory and bifurcation analysis of Equation (0.1)-(0.2) is discussed in Section 1. We further explain how singularity theory is related to *ideal membership problem* in algebraic geometry. Section 2 describes how to treat the ideal membership problem. In this direction, computational algebraic tools such as standard and Gröbner bases for ideals in three different rings, and the concept of *finite codimension* ideals are introduced. *Intrinsic ideals* and their associated ideal representations are discussed in Section 3. We further explain a procedure for computing the intrinsic part of an ideal. Section 4 gives our suggestions on how to implement some objects and results from singularity theory. These implementations include high order term ideals, tangent spaces, transition sets, persistent bifurcation diagram classifications, normal forms and the universal unfolding. The capabilities of the main features of **Singularity** along with a few examples are sketched in Section 5. Finally, our future and in-progress projects are outlined in Section 6.

## 1 Introduction

Due to the local nature of the problem (0.1)-(0.2), we recall the notion of *smooth germs* around a base point. Two maps are considered as *germ-equivalent* when both maps are locally identical; more precisely, when there exists a neighborhood of the base point so that both maps are equal on the neighborhood. A *germ* is a *germ-equivalence* class of a smooth map. We denote  $\mathcal{E}$  for the set of all scalar smooth germs whose base point is the origin. From now on we merely work with elements of  $\mathcal{E}$  rather than a scalar smooth map; see [31, 156].

Following [31, Chapter 7] and [29], we study the local zeros of maps when there is only one distinguished parameter denoted by  $\lambda$ , i.e.,

$$g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad g(x, \lambda) = 0, \quad g(0, 0) = g_x(0, 0) = 0, \quad \text{and} \quad m := 1. \quad (1.1)$$

The effect of additional parameters may be treated by study of their small perturbations. The main goal of this theory is to classify *qualitative* types of Equation (1.1) and its arbitrary small perturbations. In order to achieve this goal, we first define a *qualitative property* as a property that is invariant under an appropriate equivalence relation. Here, we use *contact-equivalence* relation:

- We say that the germs  $g$  and  $h$  are contact-equivalent when there exist a smooth germ  $S(x, \lambda)$  and diffeomorphic germs  $X$  and  $\Lambda$  such that

$$g(x, \lambda) = S(x, \lambda)h(X(x, \lambda), \Lambda(\lambda)), \quad (1.2)$$

where  $S(x, \lambda) > 0$ ,  $X_x(x, \lambda) > 0$  and  $\Lambda'(\lambda) > 0$ .

Other alternative equivalences have also been reported in the literature based on their applications; e.g., see [7, 45] where *right-left* and *strategy equivalences* have been used. *Right equivalence* is also used to study the isolated hypersurface of geometric singularities and to study the bifurcation analysis in *catastrophe theory*; see [7, 32, 44]. The contact-equivalence is here used, since it is the most natural equivalence relation preserving the zero structures.

Now we briefly describe the approach in singularity theory and how the computational algebraic tools may help to implement them. To study the local zeros of (1.1), we choose a representative (say  $f$ ) from contact-equivalent class of  $g$  that is considered to be the *simplest* for the analysis and call it a *normal form*. In order to compute the normal form of a singular germ, we need to compute certain ideals in the *local ring of smooth germs*. This signifies the importance of the well-known ideal membership problem in algebraic geometry, that is, deciding what kinds of germs belong to an ideal generated by a given set. One may study the zero structures of the normal forms and then, conclude about the solution behavior of Equation (1.1). For instance, let

$$g(x, \lambda) := \exp(x^2) + 2 \cos(x) - 3 + \sin(\lambda). \quad (1.3)$$

Using the command `Normalform(g, 5)` in `Singularity`, we obtain its normal form by  $f(x, \lambda) := \frac{7}{12}x^4 + \lambda$ . The bifurcation diagrams of  $f$  and  $g$  are depicted in Figure 1(a) and 1(b). Here, `Transformation(g, f, 5)` provides an approximation  $(\lambda + x + \lambda^2 + \lambda x, 1 + \frac{1}{6}\lambda^2 - \frac{7}{12}\lambda^3 - \frac{7}{3}\lambda^2x - \frac{7}{2}\lambda x^2 - \frac{7}{3}x^3 - \frac{833}{360}\lambda^4 - \frac{28}{3}\lambda^3x - 14\lambda^2x^2 - \frac{28}{3}\lambda x^3 - \frac{7}{3}x^4)$  modulo degree 5 for  $(X(x, \lambda), S(x, \lambda))$  which transforms  $g$  into  $f$  via Equation (1.2). The locally invertible transformation  $X(x, \lambda)$  sends the bifurcation diagram of  $f$  into that of  $g$ .

Real life problems can not be perfectly modeled by a system of equations and *imperfections* are always inevitable. Furthermore, the singularity of a germ  $g$  implies that the zeros of a small perturbation of  $g$ , say

$$h(x, \lambda, \beta) = 0, \text{ where } h(x, \lambda, 0) = g(x, \lambda), \quad (1.4)$$

may behave substantially different than what the zero structure associated with  $g = 0$  does. The parameterized germ  $h$  in (1.4) is called an *unfolding* of  $g$ . Hence, *modeling imperfections* and the possible existence of additional parameters in a model are the main obstacles of simply using normal forms of a singular germ for the qualitative understanding of a real life problem. Thus, the approach needs to be refined through the notion of *universal unfolding*. In fact, we are interested to find a parameterized family like

$$G(x, \lambda, \alpha) = 0, \quad \alpha \in \mathbb{R}^p, \quad (1.5)$$

such that for any small perturbation  $\epsilon p(x, \lambda, \epsilon)$ , the germ  $g + \epsilon p(x, \lambda, \epsilon)$  would be contact-equivalent to  $G(x, \lambda, \alpha(\epsilon))$  for some germ  $\alpha(\epsilon)$ . We call such germ  $G$  a *versal unfolding* of  $g$ . A versal unfolding with the minimum possible number of parameters is called a *universal unfolding* for  $g$ ; see [31,

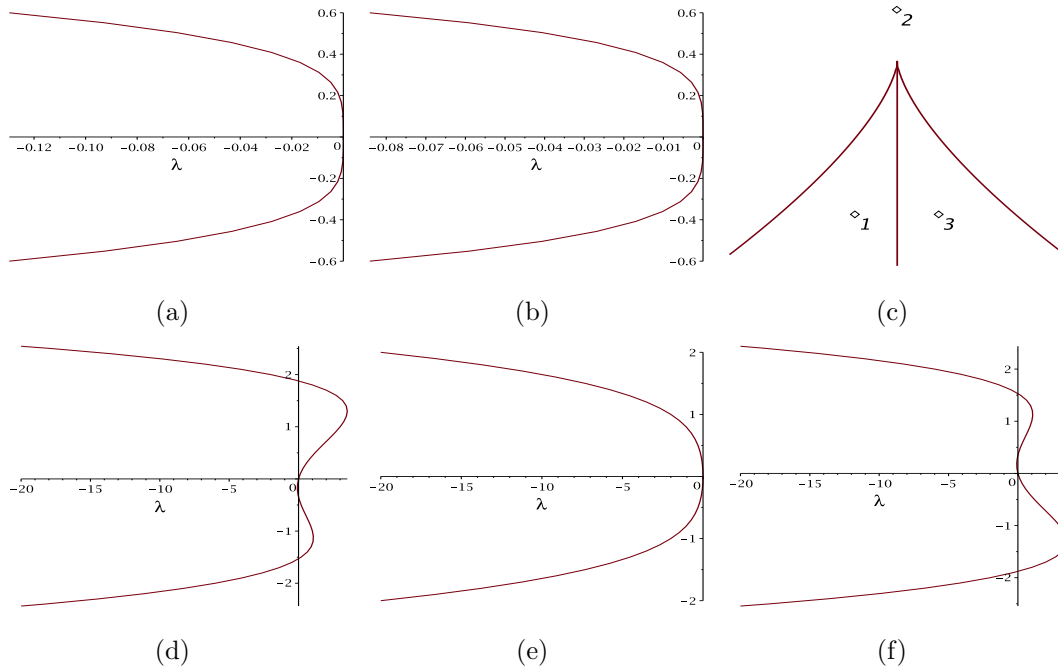


Figure 1: The vertical axes stand for the state variable  $x$ . Diagrams 1(a) and 1(b) depict bifurcation diagrams of  $g$  in Equation (1.3) and its normal form  $f$ , respectively. The diagram 1(c) shows the transition set and the regions 1-3 associated with Equation (1.4). The second row illustrates the persistent bifurcation diagrams associated with these regions.

Definitions 6.4.2 and 6.4.3, Page 176] and [29, Definitions 1.1 and 1.3, Pages 120–121]. The number of parameters in a universal unfolding is named the *codimension* of  $g$ . The universal unfolding accommodates any possible modeling imperfections, any arbitrary small perturbation and also the existence of any possible number of parameters (in addition to the distinguished parameter  $\lambda$ ). In order to derive the universal unfolding of a given singularity, the computation of a vector space called *tangent space* or instead a basis for its complement is required. Using **Singularity**'s command **UniversalUnfolding(g)** for  $g$  in (1.3) gives rise to

$$G(x, \lambda, \alpha_1, \alpha_2) := x^4 + \lambda + \alpha_1 x + \alpha_2 x^2. \quad (1.6)$$

*Bifurcation diagram classification* of the universal unfolding gives an insight to the zero structure of a germ and any of its perturbations. This is studied by the notion of *persistence* in the bifurcation diagrams. In fact a bifurcation diagram is called *persistent* when all its small perturbations remain self contact-equivalent, and otherwise it is called *nonpersistent*. Finding nonpersistent systems and their associated subset of the parameter space (the so-called *transition set*  $\Sigma$ ) play a central role in this classification. More precisely, all parametric germs associated with parameters in a connected component of  $\Sigma^c$ , complement of  $\Sigma$ , are contact-equivalent. Therefore by choosing a parameter from each connected component of  $\Sigma^c$ , a complete list of persistent bifurcation diagrams modulo contact-equivalent is obtained. The non-persistence may either originate from a local nature or be caused

by the singular boundary conditions. Local nonpersistent bifurcation diagrams are determined with families of germs associated with three semi-algebraic parameter spaces of codimension one; i.e., *bifurcation*  $\mathcal{B}$ , *hysteresis*  $\mathcal{H}$  and *double limit point*  $\mathcal{D}$ ; see [29, Page 140] for details. Nonpersistent germs associated with boundary conditions add extra complications into the solution dynamics, when bifurcation diagrams cross the boundary; see [29, Pages 154–165]. Given our description for any finite codimension singularity, the connected components of the complement set of the transition set, i.e.,  $\Sigma^c$ , provide the qualitative classification of persistent bifurcation diagrams. The command `TransitionSet( $G, [x, \lambda, \alpha_1, \alpha_2]$ )` for  $G$  in (1.6) gives  $\mathcal{B} := \emptyset$ ,  $\mathcal{H} := \{(\alpha_1, \alpha_2) \mid 8\alpha_2^3 = -27\alpha_1^2\}$ ,  $\mathcal{D} := \{(\alpha_1, \alpha_2) \mid \alpha_1 = 0, \alpha_2 \leq 0\}$ , and  $\Sigma := \mathcal{H} \cup \mathcal{D}$ . The transition set  $\Sigma$  is plotted in Figure 1(c). `PersistentDiagram( $G$ )` plots a complete list of all contact-inequivalent types of persistent bifurcation diagrams for  $G$ ; see the second row in Figure 1.

## 2 Ideal membership problem and tools from algebraic geometry

Given our description in the previous section, we mainly need to address the *ideal membership problem* in the ring of smooth germs. We call an *ideal basis* for the (finite) *generators* of a finitely generated ideal. The *ideal membership problem* refers to the question on whether or not an element belongs to a predefined ideal. Although a *finitely generated ideal* is defined by a ideal basis, yet it is not an easy task to understand whether a given element belongs to the ideal or not. There are two different ways to facilitate the ideal membership problem. One is to find a convenient ideal presentation from which the ideal membership can be easily understood. This is feasible for certain ideals and will be addressed in Section 3. The second is a computational approach based on a division algorithm. Dividing a given element by the ideal basis (generators), the element certainly belongs to the ideal when the remainder is zero. However for non-zero remainders with respect to an arbitrary ideal basis, the division may not help to conclude about the membership of the element in the ideal. B. Buchberger in 1965 introduced an ideal basis (called *Gröbner basis*) for ideals in the polynomial ring on which zero/nonzero remainders would indeed conclude the ideal membership; see [8–11].

Armbruster and Kredel [4] suggested a cognitive use of Gröbner basis for computing universal unfolding; also see [3]. Then, Gatermann and Lauterbach [19] extended the tools for equivariant bifurcation problems. Wright and Cowel in [13] noticed that a local version of Gröbner basis is indeed the appropriate tool to work with it in this theory. In fact, naive uses of Gröbner basis yield wrong results for many singularities (e.g., see Remark 2.17). The main reason is that Gröbner basis may work with polynomial germs, while the ring of smooth germs is a considerably larger ring than the polynomial germs. Recall that a set may generate a larger ideal in a ring than what it generates in a subring. Since Wright and Cowel’s remark, the only result in this direction is due



to Gattermann and Hosten [20] which is a fundamental contribution. They used (mixed) *standard basis* for “*mixed modules*” over fractional maps and multi-dimensional state variables. They did not discuss computations with smooth maps and Taylor series truncations. Therefore, their suggested algorithms are limited to treat certain (finite codimension) fractional maps. Their algorithms are useful for possible implementations in a computer algebra system to simplify fractional maps with finite codimension. However, they are not sufficient to compute *normal forms*. The reason is that they attempted neither to simplify *removable intermediate order terms* nor to address the issue facing high order terms for multi-dimensional cases (there does not yet exist a complete algebraic characterizations for high order terms); for example see [20, Example 53].

The principal ideas of Gröbner basis have been extended for localizations of the polynomial ring and formal power series. Here, we are concerned with local rings and the terminology “*Standard basis*” (not Gröbner basis) is usually used in the context of local rings. Following Wright and Cowel’s remark, Gattermann and Hosten [20] used the localizations of the polynomial ring (a local ring) to circumvent the problem. The localization of the polynomial-germ ring gives rise to the fractional germs, i.e., germs with a fractional representative in their germ-equivalent class. Although the set of all fractional germs is a local ring, yet it is still a much smaller ring than the ring of smooth germs. The other alternative is the local germ ring of all formal power series. This is a larger ring than the fractional germs and it is, perhaps, suitable due to Borel lemma. (Borel lemma indicates the existence of a one-to-one correspondence modulo flat functions between smooth germs and formal power series through their Taylor series expansion.) One of the main obstacles in working with the local rings is termination of the algorithms. There are methods in the literature for computing the *standard basis* (for fractional maps using Mora normal form) so that they solve the problem of algorithm terminations. Yet for the case of formal power series, no computer program can compute and store infinite formal power series expansions and thus, their truncations up to certain degrees are mostly unavoidable. In other words, the computations are performed modulo a sufficiently high degree (we will justify this in Theorem 2.16 (part 4)). In this section we investigate and compare the computations in the local (germ) rings of polynomials, the fractional germs and formal power series with those in the ring of smooth germs. We discuss the circumstances on which they can be alternatively used. This helps to efficiently use the different algorithms and yet ensure about correctness of the results.

## 2.1 Standard and Gröbner bases for ideals

Let  $K$  be a field of characteristic zero; in particular we are interested in the field of real numbers. For our convenience, we simply identify a given germ with a convenient representative of that germ. For instance, we talk about the polynomial germ ring over the field  $K$  and denote it by  $K[x, \alpha]$  while we mean the ring of all smooth germs whose germ-equivalent class have a polynomial representative. The quotient ring of  $\mathcal{E}$  over the ideal of all flat germs is denoted by  $K[[x, \alpha]]$  due to the fact that it

is ring-isomorphic to the ring of formal power series. Thus, we call  $K[[x, \alpha]]$  the germ ring of *formal power series*. Further, we identify members of  $K[[x, \alpha]]$  with the infinite Taylor expansion of their representative. Since  $K[x, \lambda]$  and  $K[[x, \lambda]]$  are Noetherian rings, we can guarantee termination of most algorithms during the computations.

Denote  $\deg(X^\alpha) := \alpha_1 + \dots + \alpha_n$  when  $X^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ ,  $X := (x_1, x_2, \dots, x_n)$  and  $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_n) \in Z_{\geq 0}^n$ . Any expression like  $X^\alpha$  is called a *monomial germ* while a *term* in  $\mathcal{E}$  means a monomial germ along with its coefficient. We define the lexicographic ordering  $\prec_{\text{lex}}$  on monomial germs  $X^\alpha$  as follows:

$$X^\alpha \prec_{\text{lex}} X^\beta \text{ when } \alpha_i - \beta_i \text{ is negative for } i := \inf\{j \mid \alpha_j - \beta_j \neq 0\}.$$

**Definition 2.1.** A *local order*  $\prec$  is a total ordering (every two terms are comparable) and furthermore,

- For any  $\alpha, \beta, \gamma \in Z_{\geq 0}^n$ , the condition  $X^\alpha \prec X^\beta$  implies  $X^{\alpha+\gamma} \prec X^{\beta+\gamma}$ .
- $x_i \prec 1$  for all  $i = 1, \dots, n$ . We further assume that  $0 \prec X^\alpha$  for any  $\alpha$ .

Due to Dickson's lemma (see [6, Page 251] and [14, Page 71]), every (infinite) set of monomials have a maximum with respect to any arbitrary local order; here, the condition  $X^\beta \mid X^\alpha$  implies  $X^\alpha \preceq X^\beta$ . An important example of a local order is *anti-graded lexicographic* ordering  $\prec_{\text{alex}}$  defined by

$$X^\alpha \prec_{\text{alex}} X^\beta \Leftrightarrow \begin{cases} \deg(X^\alpha) > \deg(X^\beta), \\ \deg(X^\alpha) = \deg(X^\beta) \quad \text{and} \quad X^\alpha \prec_{\text{lex}} X^\beta. \end{cases}$$

The localization of the polynomial germ ring is defined as

$$\mathcal{R} := \left\{ \frac{f}{g} \mid f, g \in K[x, \lambda], g(0, 0) \neq 0 \right\}$$

whose unique maximal ideal is generated by  $x$  and  $\lambda$ . It is common to denote  $\mathcal{R}$  with  $K[x, \lambda]_{\langle x, \lambda \rangle}$ . We call  $\mathcal{R}$  the *ring of fractional germs*. Throughout this paper, we denote

$$\mathcal{R} \text{ for either of the local rings } K[[x, \lambda]], \mathcal{R}, \text{ or } \mathcal{E},$$

unless it is explicitly stated.

**Definition 2.2.** Let  $f \in \mathcal{R}$  and  $\prec$  be a local order. The infinite jet and  $k$ -jet of  $f$  are defined by its Taylor series expansion around the origin and denoted by

$$J^\infty(f) := \sum_{(i,j)} a_{ij} x^i \lambda^j, \text{ and } J^k(f) := \sum_{i+j \leq k} a_{ij} x^i \lambda^j \quad \text{for } a_{ij} \in K.$$

The set of terms of  $f$  are defined by

$$\text{Terms}(f) := \{a_{ij}x^i\lambda^j \mid a_{ij} \neq 0\},$$

i.e., all terms appearing in  $J^\infty(f)$ . When  $\text{Terms}(f) \neq \emptyset$ , the leading term of  $f$  is

$$LT(f) = \max \text{Terms}(f),$$

i.e.,  $LT(f) \in \text{Terms}(f)$  and  $p \prec LT(f)$  for any  $p \in \text{Terms}(f) \setminus \{LT(f)\}$ . The germ  $g$  is flat iff  $\text{Terms}(f) = \emptyset$ . We define  $LT(f) := 0$ , when  $f$  is a flat germ, i.e.,  $J^\infty(f) = 0$ . The coefficient and monomial of the leading term are respectively called the *leading coefficient* ( $LC$ ) and *leading monomial* ( $LM$ ). For the case of  $f \in K[x, \lambda]$  and  $\prec_{\text{lex}}$ , we may similarly define  $LT(f)$ ,  $LC(f)$  and  $LM(f)$ .

Now we present some known definitions, terminologies and theorems from [6, 14, 15, 32]. These are suitably modified and generalized to fit in our purpose.

**Definition 2.3** (Remainder). A *remainder* of a germ  $f$  with respect to the set of germs  $G = \{g_1, \dots, g_m\} \subset \mathcal{E}$  and the local order  $\prec$  is defined as a germ  $\text{Rem}(f, G, \prec) \in \mathcal{E}$  so that

- (1)  $\text{Rem}(f, G, \prec) = f - \sum_{i=1}^m q_i g_i$ , for some  $q_1, \dots, q_m \in \mathcal{E}$  so that  $LT(q_i g_i) \preceq LT(f)$ .
- (2) No term of  $\text{Rem}(f, G, \prec)$  is divisible by any of  $LT(g_i)$  for  $i = 1, \dots, m$ .

By replacing  $\mathcal{E}$  with  $K[[x, \lambda]]$  in Definition 2.3, the remainders in the ring of  $K[[x, \lambda]]$  is readily defined; also see [6, Pages 251-252]. The same is true for the case of polynomial germ ring  $K[x, \lambda]$  provided that the local order  $\prec$  would be instead a monomial ordering like  $\prec_{\text{lex}}$ . The remainders in the ring of  $\mathcal{R}$  is defined in [15, Page 170] using *Mora normal form algorithm*. Usually, the terminology of *normal form* is used rather than *remainder*. However, we choose *remainder* as the other may cause confusion with the *normal forms* of germs in singularity theory. The division here is related to Malgrange preparation theorem and Mather division theorem; see [41, Corollaries A.6.2 and A.7.2, Theorem A.7.1].

When  $f$  is flat, its remainder with respect to any set of germs and local order is flat. The remainder is not necessarily unique even modulo flat germs; also see [6, Pages 251-252]. In fact, the remainder is unique modulo flat germs when  $G$  is a *standard basis* (or Gröbner basis for the case of  $K[x, \lambda]$ ), defined as follows.

For an ideal  $I$  in  $\mathcal{R}$ , we define the leading term ideal  $LT(I)_{\mathcal{R}}$  by

$$LT(I)_{\mathcal{R}} := \langle LT(f) : f \in I \rangle_{\mathcal{R}}.$$

**Definition 2.4** (Standard basis). Let  $I$  be an ideal in  $\mathcal{R}$  with a finite generating set  $\{g_1, \dots, g_m\} \subset I$ . The set  $\{g_1, \dots, g_m\}$  is called a *standard basis* of  $I$  when

$$LT(I)_{\mathcal{R}} = \langle LT(g_1), \dots, LT(g_m) \rangle_{\mathcal{R}}. \quad (2.1)$$

The set  $\{g_1, \dots, g_m\} \subseteq \mathcal{R}$  is called a standard basis in  $\mathcal{R}$  when it is a standard basis for the ideal  $\langle g_1, \dots, g_m \rangle_{\mathcal{R}}$ .

**Remark 2.5.** (a) The set  $\{g_1, \dots, g_m\} \subset K[x, \lambda]$  is called Gröbner basis with respect to  $\prec_{\text{lex}}$  when the condition (2.1) holds.

(b) Any finite set of germs  $\{g_i \mid 1 \leq i \leq m\}$  in  $\mathcal{E}$  is a standard basis in  $\mathcal{E}$  iff the set of formal power series  $\{J^\infty(g_i)\}$  is a standard basis in  $K[[x, \lambda]]$ .

(c) Let  $\mathcal{R} := K[x, \lambda], K[[x, \lambda]]$  or  $\mathcal{R}$  and  $\{g_1, \dots, g_m\} \subset I$  be a finite set. Then,

$$LT(I)_{\mathcal{R}} = \langle LT(g_1), \dots, LT(g_m) \rangle_{\mathcal{R}}$$

implies that  $\{g_1, \dots, g_m\} \subset I$  is a generating set for  $I$ . This has a simple argument as follows; also see [5, Page 206]. For any  $f \in I$ ,

$$r := \text{Rem}(f, \{g_i \mid i = 1, \dots, m\}, \prec) \in I.$$

Thus,  $LT(r) \in LT(I)_{\mathcal{R}}$  and  $LT(r)$  must be factored by  $LT(g_i)$  for some  $i$ . The latter implies that  $r = 0$ , otherwise this contradicts with  $r$  being a remainder.

Computation of standard basis uses the notion of  $S$ -germs. Let  $f, g \in \mathcal{R}$  and  $\preceq$  be a local order. Then,  $S$ -germ of  $f$  and  $g$  is defined by

$$S(f, g) = \begin{cases} \frac{LCM(LM(f), LM(g))}{LT(f)} f - \frac{LCM(LM(f), LM(g))}{LT(g)} g & \text{if } J^\infty(f)J^\infty(g) \neq 0, \\ 0 & \text{for } J^\infty(f)J^\infty(g) = 0. \end{cases}$$

Here  $LCM$  stands for the least common multiple for a pair of monomials.

**Theorem 2.6.** (Also see Hironaka Theorem on [6, Page 252]) Let  $\mathcal{R} = \mathcal{E}$  or  $K[[x, \lambda]]$ ,  $G = \{g_1, \dots, g_m\} \subsetneq \mathcal{R}$ ,  $0 \neq f \in \mathcal{R}$  and  $\prec$  be a local order. Then,

- (a) Always  $f$  has a remainder with respect to  $\prec$  and  $G$ .
- (b) The set  $G$  is a standard basis iff the remainder of  $f$  with respect to  $G$  and  $\prec$  is unique modulo flat germs.
- (c) (Buchberger's Criterion) The set  $G = \{g_1, \dots, g_m\}$  is a standard basis iff for all  $i, j$ , the expression  $\text{Rem}(S(g_i, g_j), G, \prec)$  is a flat germ.
- (d) The set  $G$  is a standard basis iff  $\text{Rem}(f, G, \prec)$  is flat for all  $f \in \langle G \rangle_{\mathcal{R}}$ .

*Proof.* When  $\mathcal{R} := K[[x, \lambda]]$ , the claim is *Hironaka Theorem* given on [6, Page 252]; also see [35]. Therefore, we assume that  $\mathcal{R} = \mathcal{E}$ . By Hironaka Theorem, the remainder always exists in the ring of  $K[[x, \lambda]]$ , i.e., there exists a

$$\text{Rem}(J^\infty(f), \{J^\infty(g_i)\}_{i=1}^m, \prec) \in K[[x, \lambda]]$$

and  $q_i \in K[[x, \lambda]]$  so that

$$\text{Rem}(J^\infty(f), \{J^\infty(g_i)\}_{i=1}^m, \prec) = J^\infty(f) - \sum_{i=1}^m q_i J^\infty(g_i)$$

with  $LT(J^\infty(f)) \preceq LT(q_i J^\infty(g_i))$ . Further for

$$p \in \text{Terms}\left(\text{Rem}(J^\infty(f), \{J^\infty(g_i)\}_{i=1}^m, \prec)\right),$$

$p$  is not divisible by any of  $LT(J^\infty(g_i))$  for  $i = 1, \dots, m$ . Our proposed division algorithm for germs in  $\mathcal{E}$  follows

$$\begin{aligned} \text{Rem}(f, \{g_i\}_{i=1}^m, \prec) &:= f - \sum_{i=1}^m \hat{q}_i g_i \\ &= \text{Rem}(J^\infty(f), \{J^\infty(g_i)\}_{i=1}^m, \prec) \\ &\quad + f - J^\infty(f) + \sum_{i=1}^m q_i J^\infty(g_i) - \sum_{i=1}^m \hat{q}_i g_i, \end{aligned} \tag{2.2}$$

where  $\hat{q}_i \in \mathcal{E}$ ,  $J^\infty(\hat{q}_i) := q_i$ . Note that the last line in (2.2) is merely a flat germ. Equation (2.2) implies that the division in  $\mathcal{E}$  merely differs by flat functions from its analog in  $K[[x, \lambda]]$ . The proof of part (a) is complete by Equation (2.2).

Let  $r = f - \sum_{i=1}^m q_i g_i$  satisfying (1) and (2) in Definition 2.3. Then,

$$\begin{aligned} J^\infty(r) &= J^\infty(f) - \sum_{i=1}^m J^\infty(q_i g_i) = J^\infty(f) - \sum_{i=1}^m J^\infty(q_i) J^\infty(g_i) \\ &= \text{Rem}(J^\infty(f), \{J^\infty(g_i)\}_{i=1}^m, \prec). \end{aligned}$$

Therefore,  $\text{Rem}(J^\infty(f), \{J^\infty(g_i)\}_{i=1}^m, \prec)$  is unique if and only if  $\{J^\infty(g_i)\}$  is a standard basis in  $K[[x, \lambda]]$ . This completes the proof by Remark 2.5. The proof of parts (c) and (d) are given by  $J^\infty(S(g_i, g_j)) = S(J^\infty g_i, J^\infty g_j)$ , Equation (2.2) and Remark 2.5(a).  $\square$

In order to compute the standard basis for an ideal in  $K[[x, \lambda]]$  (and also in  $\mathcal{E}$ ), one needs to sequentially enlarge and update its set of generators by only adding the *non-flat* remainders (with respect to the updated generators and a given local order) of the  $S$ -germs of the generator pairs. This process is usually known as *Buchberger algorithm* and it terminates when the ascending ideals generated by the leading terms of the updated generators stops any further enlargement. The Buchberger algorithm is finitely terminated due to the fact that  $K[[x, \lambda]]$  is a Noetherian ring.

**Example 2.7.** (a) The division of a polynomial by a set of polynomials in our division algorithm may involve formal power series; for example we have  $\text{Rem}(1, 1 - x, \prec_{\text{alex}}) = 0 = 1 - (\sum_{n=0}^{\infty} x^n)(1 - x)$ .

(b) The remainder of a polynomial divided by a polynomial may give rise to an *infinite* formal power series. For instance let  $G := \{x\lambda - x^2\lambda^2 - x^4\}$ . Then,

$$\begin{aligned} h &:= \text{Rem}(x^2\lambda, G, \prec_{\text{alex}}) = x^5 + x^9 + 2x^{13} + 5x^{17} + 14x^{21} + \dots \\ &= x^2\lambda - (x + x^2\lambda + x^5 + x^3\lambda^2 + 2x^6\lambda + x^4\lambda^3 + \dots)(x\lambda - x^2\lambda^2 - x^4). \end{aligned}$$

Since the generator of an ideal with only one generator is always a standard basis, the remainder here is unique.

(c) This example is to show that a finite set of polynomial ideal basis in  $\mathcal{E}$  may lead to a standard basis that includes non-polynomial germs. Let  $G = \{f := x\lambda - x^2\lambda^2 - x^4, g := \lambda - x\lambda - x\lambda^2 - x^3\}$ . Then,  $S(f, g) = x^2\lambda$ . It is easy to verify that  $S = \{f, g, h\}$  is a standard basis, where  $h$  is defined in part (b).

The first and second Buchberger criteria are applied for efficient computation of standard basis in **Singularity**. However, we will not discuss them in this paper.

**Definition 2.8** (Reduced standard germ basis). Let  $G = \{g_1, \dots, g_n\}$  be a standard basis and  $LC(g_i) = 1$  for  $i = 1, \dots, n$ . When

- $LT(g_i) \nmid p$  for all  $p \in \text{Terms}(g_j)$ , except for when  $p = LT(g_i)$  and  $i = j$ ,

the set  $G$  is called a *reduced standard basis*.

Given a local order  $\prec$ , [6, Theorem 2.1, Page 255] states that any ideal in  $K[[x, \lambda]]$  has a unique reduced standard basis.

**Theorem 2.9.** *With respect to any local order  $\prec$ , every finitely generated ideal  $I \subseteq \mathcal{E}$  has a reduced standard basis. The standard basis is unique modulo flat germs.*

*Proof.* Put  $\hat{I} = J^\infty(I) := \langle J^\infty f \mid f \in I \rangle$ . By [6, Theorem 2.1, Page 255], there exists a unique reduced standard basis  $\{g_1, \dots, g_m\}$  for  $\hat{I}$  where  $g_i = J^\infty(G_i)$  with  $G_i \in I$ . Since  $LT(G_i) = LT(g_i)$  for  $i = 1, \dots, m$ ,

$$LT(\hat{I})_{K[[x, \lambda]]} = \langle LT(g_i) : i = 1, \dots, m \rangle_{K[[x, \lambda]]} = \langle LT(G_i) : i = 1, \dots, m \rangle_{K[[x, \lambda]]}.$$

Now we have

$$\begin{aligned} \{LT(f) \mid f \in I\} &= \{LT(f) \mid f \in \hat{I}\} \subset \langle LT(G_i) : i = 1, \dots, m \rangle_{K[[x, \lambda]]} \\ &\subset \langle LT(G_i) : i = 1, \dots, m \rangle_{\mathcal{E}}. \end{aligned}$$

Hence,  $LT(I)_{\mathcal{E}} = \langle LT(G_i) : i = 1, \dots, m \rangle_{\mathcal{E}}$  and  $\langle LT(G_i) : i = 1, \dots, m \rangle_{\mathcal{E}} \subseteq I$ . Assuming that  $I := \langle \{f_j\}_{j=1}^n \rangle_{\mathcal{E}}$ , the set  $\{G_i\}_{i=1}^m \cup \{\text{Rem}(f_j, \{G_i\}_{i=1}^m, \prec)\}_{j=1}^n$  is a reduced standard basis, since  $\text{Rem}(f_j, \{G_i\}_{i=1}^m, \prec)$  is flat for any  $j$  due to part (c) in Theorem 2.6. The uniqueness follows [6, Theorem 2.1] and Remark 2.5 (a).  $\square$

The following Theorem along with Equation (2.2) and Buchberger algorithm provide computational guidelines on how to compute a reduced standard basis in  $\mathcal{E}$ .

**Theorem 2.10.** *Let  $I = \langle f_1, \dots, f_n \rangle_{\mathcal{E}}$ .*

- *There always exist germs  $g_1, \dots, g_m \in \mathcal{E}$  for  $m \leq n$  so that*

$$I = \langle g_1, \dots, g_m \rangle_{\mathcal{E}} \quad (2.3)$$

*and  $LT(g_i) \nmid p$  for all  $p \in \text{Terms}(g_j)$ , except for when  $i = j$  and  $p = LT(g_i)$ .*

- *Furthermore, assume that  $\{f_i\}_{i=1}^n$  is a standard basis in  $\mathcal{E}$ . Then, the set  $\{g_i\}_{i=1}^m$  (for  $m \leq n$ ) can be chosen so that it is a reduced standard basis.*

*Proof.* The proof here is similar to the idea on [6, Page 255] and gives us an algorithm for reduced standard basis computation. Without loss of generality assume that  $LT(f_i) \nmid LT(f_j)$  for  $i \neq j$ ; otherwise,  $f_j = \sum_{i \neq j} q_i f_i + r_j$  for some  $r_j, q_i \in \mathcal{E}$  satisfying conditions in Definition 2.3. Thus,  $f_j$  may be replaced by  $r_j$  and this may cause  $m < n$  for when  $r_j = 0$ . By Theorem 2.6, there exist  $q_{ij} \in \mathcal{E}$  for  $j = 1, \dots, n$  so that

$$r_i = \text{Rem}(f_i - LT(f_i), \{f_j\}_{j=1}^n, \prec) = f_i - LT(f_i) - \sum q_{ij} f_j.$$

Now for each  $p \in \text{Terms}(r_i)$  we have  $LT(f_j) \nmid p$ . Define  $g_i = LT(f_i) + r_i$ . This shows  $g_i \in I$  and the first claim follows. The second claim is valid due to  $LT(f_i) = LT(g_i)$ .  $\square$

**Example 2.11.** Let  $f$  be a non-zero flat germ, and  $I = \langle G \rangle_{\mathcal{E}}$  where

$$G := \{g_1 := \lambda - \lambda \exp(x), g_2 := x - \sin(x), g_3 := \lambda x + \lambda^3 + \lambda^2 \ln(1+x) + f\}.$$

Since  $LT(G) = \{\lambda x, x^3\}$  and  $LT(I) = \langle \lambda x, x^3, \lambda^3 \rangle_{\mathcal{E}}$ ,  $G$  is not a standard basis with respect to  $\prec_{\text{alex}}$ . Due to

$$\text{Rem}(S(g_1, g_2), G, \prec_{\text{alex}}) = 0, \quad \text{Rem}(S(g_2, g_3), G, \prec_{\text{alex}}) = -x^2 f$$

and  $\text{Rem}(S(g_1, g_3), G, \prec_{\text{alex}}) = \lambda^3 + f$ , we add  $\lambda^3 + f$  to the basis, i.e.,  $S_1 := \{g_1, g_2, g_3, \lambda^3 + f\}$ . Now  $S_1$  is a standard basis, yet it is not a reduced standard basis. Given Theorem 2.10 and its proof,

$$LT(g_1) \mid LT(g_3), \quad \text{Rem}(g_3, S_1 \setminus \{g_3\}, \prec_{\text{alex}}) = 0, \quad S_2 := \{g_1, g_2, \lambda^3 + f\} \text{ and}$$

$$\text{Rem}(\lambda x + g_1, S_2, \prec_{\text{alex}}) = \text{Rem}(g_2 - \frac{x^3}{6}, S_2, \prec_{\text{alex}}) = 0,$$

we define

$$S_3 := \{\lambda x, x^3, \lambda^3 + f\}.$$

Now  $S_3$  is a reduced standard basis.

## 2.2 Finite codimension

Finite codimensional ideals demonstrate an important role in ideal presentation in Section 3. Let  $\mathcal{M}_{\mathcal{E}} := \langle x, \lambda \rangle_{\mathcal{E}}$ ,  $\mathcal{M}_{\mathcal{R}} = \langle x, \lambda \rangle_{\mathcal{R}}$ ,  $\mathcal{M}_{K[[x, \lambda]]} = \langle x, \lambda \rangle_{K[[x, \lambda]]}$ , and  $\mathcal{M}_{K[x, \lambda]} = \langle x, \lambda \rangle_{K[x, \lambda]}$ .

**Definition 2.12.** An ideal  $I$  in  $\mathcal{E}$  (or in  $\mathcal{R}$ ,  $K[[x, \lambda]]$ ) is said to have a *finite codimension* when  $\mathcal{M}_{\mathcal{E}}^k \subseteq I$  ( $\mathcal{M}_{\mathcal{R}}^k \subseteq I$ ,  $\mathcal{M}_{K[[x, \lambda]]}^k \subseteq I$ ) for  $k \in \mathbb{N}$ . Equivalently,  $I$  has a complement (vector) subspace in  $\mathcal{E}$  with finite dimension.

Now we compare the ideals in  $K[x, \lambda]$  with those in  $\mathcal{R}$ ,  $K[[x, \lambda]]$  and  $\mathcal{E}$ .

**Theorem 2.13.** *Let  $I$  be a finite codimensional ideal in  $\mathcal{E}$ .*

- (1) *The ideal  $I$  has a unique reduced standard polynomial germ basis.*
- (2) *Assume that  $G := \{g_i\}_{i=1}^n$  is a standard basis for  $I$  and  $f \in \mathcal{E}$ . Then, the remainder is always a polynomial germ, i.e.,  $\text{Rem}(f, G, \prec_{\text{alex}}) \in K[x, \lambda]$ . In particular, we have  $\deg(\text{Rem}(f, G, \prec_{\text{alex}})) \leq k$  when  $\mathcal{M}_{\mathcal{E}}^{k+1} \subseteq I$ .*

*Proof.* By Theorem 2.9, we assume that  $G := \{f_i\}_{i=1}^n$  is a reduced standard basis for  $I$ . Assume that  $\mathcal{M}_{\mathcal{E}}^{k+1} \subseteq I$  for some  $k$ . Then, for any  $0 \leq l \leq k+1$  there exists  $f_i \in G$  so that  $LT(f_i)|x^{k+1-l}\lambda^l$ . On the other hand for any  $p \in \text{Terms}(f_j - LT(f_j))$  and  $j \leq n$ ,  $LT(f_i) \nmid p$  and hence,  $x^{k+1-l}\lambda^l \nmid p$ . The latter implies that  $J^\infty(f_j) = J^{k+1}(f_j)$  for all  $j$ . Finally, we recall that any finite codimension ideal includes the set of all flat germs. This allows us to replace  $f_j$  with  $g_j := J^{k+1}(f_j)$  and  $S := \{g_j\}_{j=1}^n$  is a reduced standard basis. The second claim similarly follows from  $\mathcal{M}_{\mathcal{E}}^{k+1} \subseteq LT(I)_{\mathcal{E}} = \langle LT(g_i) \rangle_{\mathcal{E}}$  and the fact that no term of  $\text{Rem}(f, G, \prec_{\text{alex}})$  is divisible by  $LT(g_i)$ .  $\square$

**Example 2.14.** Let  $S = \{f_i\}_{i=1}^n$  be a reduced standard basis whose generated ideal is of finite codimension. Then,  $\text{Rem}(f, S, \prec_{\text{alex}}) = f - \sum q_i f_i \in K[x, \lambda]$ . However,  $q_i$  may not always be a polynomial germ. Consider the ordering  $\lambda \prec_{\text{alex}} x$  and define

$$I := \langle g_1 := 2\lambda^3 - 3\lambda^2x + x^5, g_2 := -3x\lambda^2 + 5x^5, g_3 := -3\lambda^3 + 5x^4\lambda \rangle.$$

The ideal  $I$  is finite codimensional since  $\mathcal{M}_{\mathcal{E}}^6 \subset I$ . Here,  $G = \{g_1, g_2, g_3, \frac{4}{3}x^5 - \frac{10}{9}x^4\lambda\}$  is a standard basis and

$$S = \left\{ f_1 := x\lambda^2 - \frac{25}{18}x^4\lambda, f_2 := -\frac{1}{3}g_3, f_3 := x^5 - \frac{5}{6}x^4\lambda \right\}$$

is a reduced standard basis. Now we have

$$x^3\lambda^3 = x^3f_2 + \frac{25}{18}x^5qf_1 + \frac{5}{3}\lambda x^2qf_3,$$

where  $q := \sum_{i=0}^{\infty} \left(\frac{125}{108}\right)^i x^{2i}$ ,  $\text{Rem}(x^3\lambda^3, S, \prec_{\text{alex}}) = 0$  while  $q_1 := \frac{25}{18}x^5q$  and  $q_2 := \frac{5}{3}\lambda x^2q$  are not polynomial germs.



The following theorem enables us to compute the standard basis for ideals in  $\mathcal{E}$  through computations in the fractional germs.

**Theorem 2.15.** *Consider  $I = \langle p_1, \dots, p_n \rangle_{\mathcal{E}}$ ,  $\prec_{\text{alex}}$  and  $\mathcal{M}_{\mathcal{R}}^k \subseteq \langle p_1, \dots, p_n \rangle_{\mathcal{R}} \subseteq \mathcal{R}$ . Further, suppose that  $\{q_1, \dots, q_m\}$  is a standard basis for  $\langle p_1, \dots, p_n \rangle_{\mathcal{R}}$ . Then,  $\{q_1, \dots, q_m\}$  is a standard basis for  $I$ .*

*Proof.* See Appendix 7.1.  $\square$

The following theorem is one of our main contributions in this paper and provides important alternatives and criteria for our computations in different circumstances including ideals with infinite codimension.

**Theorem 2.16.** (1) *Suppose that  $G := \{g_1, \dots, g_n\} \subset \mathcal{R}$ . Then, the ideal  $\langle g_1, \dots, g_n \rangle_{\mathcal{E}}$  has a finite codimension iff  $\langle g_1, \dots, g_n \rangle_{\mathcal{R}}$  is a finite codimension ideal. Assuming that  $\langle g_1, \dots, g_n \rangle_{\mathcal{R}}$  has a finite codimension, for nonnegative integers  $i$  and  $j$ ,  $\mathcal{M}_{\mathcal{E}}^i \langle \lambda^j \rangle_{\mathcal{E}} \subseteq \langle g_1, \dots, g_n \rangle_{\mathcal{E}}$  iff  $\mathcal{M}_{\mathcal{R}}^i \langle \lambda^j \rangle_{\mathcal{R}} \subseteq \langle g_1, \dots, g_n \rangle_{\mathcal{R}}$ .*

(2) *Let  $\mathcal{M}_{K[x, \lambda]}^k \subseteq \langle g_1, \dots, g_n \rangle_{K[x, \lambda]} \subseteq K[x, \lambda]$ . Then,  $\mathcal{M}_{K[x, \lambda]}^i \langle \lambda^j \rangle_{K[x, \lambda]} \subseteq \langle g_1, \dots, g_n \rangle_{K[x, \lambda]}$  iff  $\mathcal{M}_{\mathcal{E}}^i \langle \lambda^j \rangle_{\mathcal{E}} \subseteq \langle g_1, \dots, g_n \rangle_{\mathcal{E}}$ .*

(3) *For a finite codimension ideal  $\langle g_1, \dots, g_n \rangle_{\mathcal{R}}$  in  $\mathcal{R}$  and an ideal  $I$  in  $\mathcal{E}$ ,*

$$I = \langle g_1, \dots, g_n \rangle_{\mathcal{E}} \quad \text{iff} \quad I \cap \mathcal{R} = \langle g_1, \dots, g_n \rangle_{\mathcal{R}}.$$

*In particular, let  $I, J$  be two ideals in  $\mathcal{E}$ ,  $I$  have a finite codimension and  $I \cap \mathcal{R} = J \cap \mathcal{R}$ . Then,  $I = J$ .*

(4) *For an ideal  $I$  in  $\mathcal{E}$ , the following three conditions are equivalent.*

$$(i) \mathcal{M}_{\mathcal{E}}^k \subseteq I, \quad (ii) \mathcal{M}_{\mathcal{R}}^k \subseteq I \cap \mathcal{R}, \quad (iii) \mathcal{M}_{K[x, \lambda]}^k \subseteq I \cap K[x, \lambda].$$

(5) *Let  $g_1, \dots, g_n \in \mathcal{E}$  and  $k \leq N$ , for  $k, N \in \mathbb{N}$ . Then, either of the conditions*

$$\mathcal{M}_{\mathcal{R}}^k \subseteq \langle J^N g_1, \dots, J^N g_n \rangle_{\mathcal{R}} \quad \text{and} \quad \mathcal{M}_{K[[x, \lambda]]}^k \subseteq \langle J^N g_1, \dots, J^N g_n \rangle_{K[[x, \lambda]]},$$

*is equivalent to  $\mathcal{M}_{\mathcal{E}}^k \subseteq \langle g_1, \dots, g_n \rangle_{\mathcal{E}}$ .*

(6) *Consider the finite sequence of nonnegative integers  $k_i, l_i \in \mathbb{N}$  so that the sequence  $k_i + l_i$  is decreasing and  $l_i$  is increasing. Let  $\langle f_1, f_2, \dots, f_n \rangle_{\mathcal{E}}$  be an ideal that is not necessarily of finite codimension and*

$$I := \sum \mathcal{M}_{\mathcal{R}}^{k_i} \langle \lambda^{l_i} \rangle_{\mathcal{R}} \subseteq \langle J^N f_1, \dots, J^N f_n \rangle_{\mathcal{R}}. \quad (2.4)$$

*Then, either of the following conditions*

$$- \text{ for each } j \leq n, \text{ Terms}(f_j - J^{p_j} f_j) \subseteq \mathcal{M}_{\mathcal{R}} I \text{ for some } p_j \leq N.$$

– for each  $j \leq n$ ,  $f_j \in \mathcal{R}$  and  $f_j - J^{p_j} f_j \in \mathcal{M}_{\mathcal{R}} I$  for some  $p_j \leq N$ .

implies that

$$\sum \mathcal{M}_{\mathcal{E}}^{k_i} \langle \lambda^{l_i} \rangle_{\mathcal{E}} \subseteq \langle f_1, f_2, \dots, f_n \rangle_{\mathcal{E}}. \quad (2.5)$$

*Proof.* Part (1). The *if* part is trivial. Thus, we assume that  $\mathcal{M}_{\mathcal{E}}^k \subseteq \langle g_1, \dots, g_n \rangle_{\mathcal{E}}$  and prove that  $\mathcal{M}_{\mathcal{R}}^k \subseteq \langle g_1, \dots, g_n \rangle_{\mathcal{R}}$ . Using Nakayama lemma [29, Lemma 5.3, Page 71], it is enough to verify that  $\mathcal{M}_{\mathcal{R}}^k \subset \langle g_1, \dots, g_n \rangle_{\mathcal{R}} + \mathcal{M}_{\mathcal{R}}^{k+1}$ . Since  $\mathcal{M}_{\mathcal{E}}^k \subset \langle g_1, \dots, g_n \rangle_{\mathcal{E}}$ , for some  $a_s \in \mathcal{E}$  we have

$$x^l \lambda^{k-l} = \sum_{s=1}^n a_s g_s = J^k \sum_{s=1}^n a_s g_s = \sum_{s=1}^n J^k (J^k(a_s) g_s) = J^k \sum_{s=1}^n b_s g_s = \sum_{s=1}^n b_s g_s + r,$$

where  $a_s \in \mathcal{E}$ ,  $b_s := J^k(a_s)$  and  $r \in \mathcal{M}_{\mathcal{R}}^{k+1}$ . For the second claim, we do not need Nakayama lemma. We assume that  $\mathcal{M}_{\mathcal{R}}^k \subseteq \langle g_1, \dots, g_n \rangle_{\mathcal{R}}$ . Similar to above, for  $i+j < k$  we have  $x^l \lambda^{i+j-l} = J^{k-1}(x^l \lambda^{i+j-l}) = \sum_{s=1}^n b_s g_s + r$ , where  $r \in \mathcal{M}_{\mathcal{R}}^k$  and  $b_s \in \mathcal{R}$ .

Part (2). Proof is similar to the proof of the second claim in part (1).

Part (3). Assume that  $I = \langle g_1, \dots, g_n \rangle_{\mathcal{E}}$ . Trivially,

$$\langle g_1, \dots, g_n \rangle_{\mathcal{R}} \subseteq I \cap \langle g_1, \dots, g_n \rangle_{\mathcal{R}}.$$

Let  $f \in I \cap \mathcal{R}$ . Thus,  $f = \sum a_i g_i$  for some  $a_i \in \mathcal{E}$ . So,  $f = J^k \sum J^k(a_i) g_i + h$  for an  $h \in \mathcal{M}_{K[x, \lambda]}^{k+1}$ . Since the left hand side belongs to  $\langle g_1, \dots, g_n \rangle_{\mathcal{R}}$ , the proof of the *if* part is complete. Now assume that  $I \cap \mathcal{R} = \langle g_1, \dots, g_n \rangle_{\mathcal{R}}$  and  $f \in I$ . By part (1) and  $\langle g_1, \dots, g_n \rangle_{\mathcal{E}} \subseteq I$ ,  $J^k(f) \in I$  for some  $k \in \mathbb{N}$ . Since  $J^k(f) \in I \cap \mathcal{R}$ ,

$$J^k(f) = \sum a_i g_i, \text{ for } a_i \in \mathcal{R}.$$

On the other hand  $f - J^k(f) \in \langle g_1, \dots, g_n \rangle_{\mathcal{E}}$ . This completes the proof of part (3). Part (4) is trivial.

Part (5). Now the assumption  $\mathcal{M}_{\mathcal{R}}^k \subset \langle J^N g_1, \dots, J^N g_n \rangle_{\mathcal{R}}$  implies that for some  $a_i \in \mathcal{R}$ , and any  $i = 1, \dots, k$ ,

$$x^{k-i} \lambda^i = J^k \sum_{i=1}^n a_i J^N(g_i) = J^k \sum_{i=1}^n a_i g_i = \sum_{i=1}^n a_i g_i + r, \quad \text{for } k \leq N,$$

where  $r \in \mathcal{M}_{\mathcal{R}}^{k+1}$ . This and Nakayama lemma conclude that  $\mathcal{M}_{\mathcal{E}}^k \subseteq \langle g_1, \dots, g_n \rangle_{\mathcal{E}}$ . The converse and rest of the proof use similar arguments. Note that Nakayama lemma given in [29, Lemma 5.3] is also true when  $\mathcal{E}$  is replaced with  $\mathcal{R}$  and  $K[[x, \lambda]]$ .

Part (6) is concluded by Nakayama lemma and the fact that  $x^{k_i-m} \lambda^{l_i+m} = \sum a_i J^N f_i = \sum a_i f_i + r$ , where  $r \in \langle \text{Terms}(f_j - J^N f_j) \rangle_{\mathcal{R}} \subseteq \langle \text{Terms}(f_j - J^{p_j} f_j) \rangle_{\mathcal{R}} \subset \mathcal{M}_{\mathcal{R}} I$ .  $\square$

The hypothesis of part (1) in Theorem 2.16 allows us to use fractional germs for computations while the condition in part (2) permits the use of Gröbner basis.

**Example 2.17.** Part (1) from the previous theorem is not valid when  $\mathcal{R}$  is replaced with  $K[x, \lambda]$ . For instance, consider the example given in [38, Table 1, III.1 for  $k = 5$ ] and [29, Page 77],

$$I := \langle g_1 := x^5 + x^3\lambda + \lambda^2, g_2 := 5x^5 + 3x^3\lambda, g_3 := 5x^4\lambda + 3x^2\lambda^2 \rangle_{\mathcal{E}}.$$

The ideal  $I$  has a finite codimension since  $\mathcal{M}_{\mathcal{E}}^6 \subset I$ ; see [29, Page 77]. However, the ideal  $I_0 := \langle g_1, g_2, g_3 \rangle_{K[x, \lambda]}$  has an infinite codimension in  $K[x, \lambda]$ . The reason for this is as follows. The reduced Gröbner basis of  $I$  with respect to  $\lambda \prec_{\text{lex}} x$  is given by

$$G := \{3125\lambda^3 + 108\lambda^4, 18\lambda^3 + 125\lambda^2x, 2x^3\lambda + 5\lambda^2, 2x^5 - 3\lambda^2\}.$$

Further for any natural number  $n \geq 3$ , the remainder  $\text{Rem}(x^n, G, \prec_{\text{lex}}) = c\lambda^3$  where  $c \in K$ . Therefore,  $x^n$  does not belong to  $I_0$  and  $I_0$  is an infinite codimensional ideal.

### 3 Finite codimension ideal representation using intrinsic ideals

An elegant approach for answering the ideal membership problem is to provide a good presentation for ideals. In this section we define intrinsic ideals and use it for such representation. This plays a central role in developing **Singularity** and is a prerequisite for most computations of objects presented in Section 4.

**Definition 3.1.** When only the trivial reparametrization  $\Lambda(\lambda) = \lambda$  is allowed in Equation (1.2), the associated relation  $\sim_s$  is called *strongly equivalent relation*; see [29, Page 51]. Then, an ideal  $I$  is called *intrinsic* [29, Page 81] when  $I$  includes all strongly equivalent classes of its members.

By [29, Proposition 7.1], every finite codimension ideal  $I$  in  $\mathcal{E}$  is intrinsic if and only if there exist nonnegative integers  $s, m_i, n_i$  for  $i = 0, \dots, s$  so that

$$I = \sum_{i=0}^s \mathcal{M}_{\mathcal{E}}^{m_i} \langle \lambda^{n_i} \rangle_{\mathcal{E}}, \quad (3.1)$$

$n_0 = 0$ , and the sequence  $n_i$  is strictly increasing while  $m_i + n_i$  is strictly decreasing. The conditions on  $m_i$  and  $n_i$  make the representation (3.1) unique. Equation (3.1) for intrinsic ideals gives a convenient answer for the ideal membership problem. The monomials  $x^{m_i} \lambda^{n_i}$  for  $i = 0, \dots, s$  are called *intrinsic generators* of the intrinsic ideal  $I$ . For non-intrinsic ideals or more generally for a vector space  $I$ , we define their intrinsic part; i.e., we denote  $\text{Itr}(I)$  for the largest intrinsic ideal contained in  $I$ .

**Lemma 3.2.** *For an intrinsic ideal  $I$ , there always exists a reduced monomial standard basis for  $I$  that includes its intrinsic generators. Let  $I$  be a finite codimension vector subspace of  $\mathcal{E}$  or be*

a finitely generated ideal. Then, there exist nonnegative integers  $m$  and  $n$ , a reduced monomial standard basis  $\{f_j\}_{j=1}^n$  for  $\text{Itr}(I)$  and  $\{g_i\}_{i=1}^m \subset \mathcal{E}$  so that

$$I = \text{Itr}(I) + \langle \{g_i\}_{i=1}^m \rangle_{\mathcal{R}}, \quad (3.2)$$

where  $\mathcal{R} := \mathcal{E}$  or  $\mathcal{R} := K$ . Here, none of the terms in  $\text{Terms}(g_i)$  is divisible by  $f_j$ , i.e.,  $\text{Itr}(I) \cap (\cup_{i=1}^m \text{Terms}(g_i)) = \emptyset$ .

*Proof.* Let the intrinsic ideal  $I$  be given by (3.1) and  $G$  be the set of its intrinsic generators. Then for  $i = 0, \dots, s$ , we consecutively update  $G$  by adding the monomials of type  $x^\alpha \lambda^\beta$  to  $G$  where  $\alpha + \beta = m_i + n_i$ ,  $\beta \geq n_i$ , and  $x^\alpha \lambda^\beta$  is not divisible by the elements of  $G$ . This gives rise to a reduced standard monomial basis for  $I$ . For the second part, we divide the generators of  $I$  by the reduced standard monomial basis of  $\text{Itr}(I)$  and define the nonzero remainders as  $g_i$ .  $\square$

A refinement of the decomposition (3.2) for finite codimension ideals is given as follows. We denote  $I^\perp$  for the set of all monomials not in  $I$ , while  $\langle I^\perp \rangle_K$  stands for the vector space generated by  $I^\perp$ . Now by [29, Corollary 7.4], we have

$$I = \text{Itr}(I) \oplus \left( I \cap \langle \text{Itr}(I)^\perp \rangle_K \right). \quad (3.3)$$

### 3.1 Computation

In this section we describe our suggestion on how to compute the intrinsic ideal representation (3.1) and next, the representation (3.3) for finite codimensional ideals. The first step here is to find a lower and upper bound for the maximum natural number  $k$  so that  $\mathcal{M}_{\mathcal{E}}^k \subseteq I \triangleleft \mathcal{E}$ .

For any ideal  $J$  in a ring  $\mathcal{R}$  we define

$$\varphi_{u,J} : \frac{\mathcal{R}}{J} \longrightarrow \frac{\mathcal{R}}{J}, \quad \varphi_{u,J}(f + J) := uf + J, \quad (3.4)$$

for  $u \in \{x, \lambda\}$ . Obviously for

$$J := I \text{ and } \mathcal{R} := \mathcal{E},$$

$\mathcal{E}/I$  is a finite dimensional vector space and  $\varphi_{u,I}$  is a linear nilpotent map of nilpotent degree  $N_u \leq k$ . Therefore, we have

$$\max\{N_x, N_\lambda\} \leq k < N_x + N_\lambda.$$

Now we explain how to derive  $N_x$  and  $N_\lambda$ . Computing a (reduced) standard basis for  $I$ , say  $S = \{g_i, i = 1, \dots, n\}$ , we now introduce a vector space basis generator

$$B = \{w \mid w \text{ is a monomial, } w \notin \langle LT(g_i) \mid g_i \in S \rangle_{\mathcal{R}}\}. \quad (3.5)$$

for  $\mathcal{E}/I$ ; also see [14, Pages 177–179 and Proposition 4] and [3, Pages 128–129].

**Lemma 3.3.** *Let  $I$  be a finite codimension ideal in  $\mathcal{E}$  and  $S$  be a standard basis for  $I$ . Then,  $B + I$  is a  $K$ -vector space basis for  $\mathcal{E}/I$ .*

*Proof.* Since  $I$  is a finite codimensional ideal, is so  $LT(I)_{\mathcal{E}}$ . Thus,  $B$  is a finite set. Assume that there exists  $\sum_{i=1}^m a_i w_i \in I$  for some  $a_i \in K$ , where  $w_i \in B$ . Since  $S$  is a standard basis,  $LT(f) | LT(\sum_{i=1}^m a_i w_i)$  for some  $f \in S$ . However,  $LT(\sum_{i=1}^m a_i w_i) \in \{a_i w_i\}_{i=1}^m$ . This contradicts with (3.5), unless  $a_i = 0$  for all  $i$ . This concludes that  $B$  is  $K$ -linearly independent.

For any  $g \in \mathcal{E}$  we have  $g + I = \text{Rem}(g, S, \prec_{\text{alex}}) + I$ ,

$$\text{Terms}(\text{Rem}(g, S, \prec_{\text{alex}})) \cap LT(I)_{\mathcal{E}} = \emptyset,$$

and  $\text{Rem}(g, S, \prec_{\text{alex}})$  is a polynomial germ. This concludes the proof.  $\square$

Lemma 3.3 readily provides a matrix representation (and thus, its nilpotent degree) for  $\varphi_{u,I}$ . The natural number  $N$ , satisfying  $\max\{N_x, N_\lambda\} \leq N \leq N_x + N_\lambda - 1$ , is consecutively increased to obtain  $k = m_0$ . In fact, the remainders of  $x^i \lambda^{N-i}$  divided by the standard basis  $S$  conclude the result, thanks to Theorem 2.6 (part d).

Next we look for the maximum values of  $m_i$  and  $n_i$  such that  $\mathcal{M}_{\mathcal{E}}^{m_i} \langle \lambda^{n_i} \rangle_{\mathcal{E}} \subseteq I$ . Our suggestion uses the concept of colon ideals. Let  $J_1$  and  $J_2$  be two ideals in  $\mathcal{E}$ . We define the *colon ideal* of  $J_1$  by  $J_2$  (see [14, Definition 5, Page 194]) as

$$J_1 : J_2 = \langle f \in \mathcal{E} \mid f J_2 \subseteq J_1 \rangle_{\mathcal{E}}.$$

By an inductive procedure and repeating the above for the colon ideal  $I : \langle \lambda^n \rangle_{\mathcal{E}}$ , we obtain  $m_i$  and  $n_i$  as desired. (Note that  $n_{i-1} < n_i$  and  $m_i + n_i < m_{i-1} + n_{i-1} < m_0$  for any  $i$ .) Therefore, the only remaining challenge is to compute the maximum value  $m$  so that  $\mathcal{M}_{\mathcal{E}}^m \subseteq I : \langle \lambda^n \rangle_{\mathcal{E}}$  or equivalently,  $\mathcal{M}_{\mathcal{E}}^m \langle \lambda^n \rangle_{\mathcal{E}} \subseteq I$ . The following lemma and its' follow up comments facilitate the computations of a finite generating set for colon ideals and next, their (reduced) standard basis readily determines  $m$ . The procedure mentioned above (using Equation (3.4) and  $J := I : \langle \lambda^n \rangle_{\mathcal{E}}$ ) provides suitable lower and upper bound for  $m$ .

**Lemma 3.4.** *Suppose that  $I$  is a finite codimensional ideal in  $\mathcal{E}$ ,  $J := I \cap K[x, \lambda]$  and  $J = \langle g_1, \dots, g_n \rangle_{K[x, \lambda]}$ . Then,  $I = \langle g_1, \dots, g_n \rangle_{\mathcal{E}}$ . Further, let  $J \cap \langle g \rangle_{K[x, \lambda]} = \langle h_1, \dots, h_n \rangle_{K[x, \lambda]}$  for a monomial germ  $g \in K[x, \lambda]$ . Then,*

$$I : \langle g \rangle_{\mathcal{E}} = \left\langle \frac{h_1}{g}, \dots, \frac{h_n}{g} \right\rangle_{\mathcal{E}}. \quad (3.6)$$

*Proof.* Let  $f \in I$ . Since  $I$  has a finite codimension,  $\mathcal{M}_{\mathcal{E}}^{k+1} \subseteq I$  for some  $k$ . Hence,  $J^k(f) \in I \cap K[x, \lambda] = J$ . Theorem 2.16 (part 3) completes the proof of the first part.

For any  $u \in I : \langle g \rangle_{\mathcal{E}}$ , we have  $u \langle g \rangle_{\mathcal{E}} \subseteq I$ . Thus,

$$ug \in I \cap \langle g \rangle_{\mathcal{E}}. \quad (3.7)$$

For any  $p \in I \cap \langle g \rangle_{\mathcal{E}} \cap K[x, \lambda]$ ,  $p = fg = J^{\deg p - \deg g}(f)g$  for some  $f \in \mathcal{E}$ . Thereby,

$$I \cap \langle g \rangle_{\mathcal{E}} \cap K[x, \lambda] = J \cap \langle g \rangle_{K[x, \lambda]} = \langle h_1, \dots, h_n \rangle_{K[x, \lambda]}.$$

By the first part, we have

$$I \cap \langle g \rangle_{\mathcal{E}} = \langle h_1, \dots, h_n \rangle_{\mathcal{E}}.$$

This and Equation (3.7) imply that  $u \in \langle \frac{h_1}{g}, \dots, \frac{h_n}{g} \rangle_{\mathcal{E}}$  and  $I : \langle g \rangle_{\mathcal{E}} \subseteq \langle \frac{h_1}{g}, \dots, \frac{h_n}{g} \rangle_{\mathcal{E}}$ .

Now let  $u \in \langle \frac{h_1}{g}, \dots, \frac{h_n}{g} \rangle_{\mathcal{E}}$ . Hence,  $aug \in \langle h_1, \dots, h_n \rangle_{\mathcal{E}}$  for any  $a \in \mathcal{E}$ . On the other hand,  $\langle h_1, \dots, h_n \rangle_{\mathcal{E}} \subseteq I$  shows that  $aug \in I$  and eventually  $u \in I : \langle g \rangle_{\mathcal{E}}$ .  $\square$

Given Lemma 3.4, we would merely need to compute  $J \cap \langle g \rangle_{K[x, \lambda]}$  for the monomials  $g := \lambda^n$ . Note that the above lemma is a slightly modified version of [14, Theorem 11, Page 196] and [15, Theorem 5.5, Page 185]. We simply use  $J \cap \langle g \rangle_{K[x, \lambda]}$  rather than using the naive and alternative choice  $I \cap \langle g \rangle_{\mathcal{E}}$ . The reason for this is that the computation of  $J \cap \langle g \rangle_{K[x, \lambda]}$  using the lexicographic ordering is efficient and classic (described below) while such an approach for  $I \cap \langle g \rangle_{\mathcal{E}}$  does not seem easy to work with standard basis with local orderings.

Finally, we recall the classical approach on the intersection computation of two ideals  $J_1$  and  $J_2$  in the polynomial germ ring  $K[x, \lambda]$ ; see [14, Page 187] for more details. Let  $J_1 := \langle f_i \mid i = 1, \dots, n_1 \rangle_{K[x, \lambda]}$  and  $J_2 := \langle g_i \mid i = 1, \dots, n_2 \rangle_{K[x, \lambda]}$ . We define

$$J_3 := \langle tf_i, (1-t)g_j \mid \text{for } i = 1, \dots, n_1 \text{ and } j = 1, \dots, n_2 \rangle_{K[x, \lambda, t]}. \quad (3.8)$$

Thus,  $J_3 \cap K[x, \lambda] = J_1 \cap J_2$ . We may compute a Gröbner basis for  $J_3$  with respect to  $\prec_{\text{lex}}$  (where  $x, \lambda \prec_{\text{lex}} t$ )

$$G_3 := \{p_i(x, \lambda), q_j(x, \lambda, t) \mid \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, m\},$$

here  $p_i$  represents those basis elements independent of  $t$  while  $q_j$  represents those explicitly depending on  $t$ . Since  $\langle LT(G_3) \rangle_{K[x, \lambda, t]} = LT(J_3)$ , we may conclude that

$$G := \{p_i(x, \lambda) \mid \text{for } i = 1, \dots, n\} \quad (3.9)$$

is a Gröbner basis for  $J_1 \cap J_2$ . The reason is as follows. For any  $u \in LT(J_1 \cap J_2) \subseteq LT(J_3)$ , either  $LT(p_i)$  or  $LT(q_j)$  for some  $i$  or  $j$  must divide  $u$ . Since  $t$  divides the leading term of  $q_j$  for any  $j$ ,  $u \in \langle LT(G) \rangle_{K[x, \lambda]}$ .

Given Equation (3.3), we merely need to use the generators of  $I$ , a standard basis for  $\text{Itr}(I)$  and Theorem 2.6 (part d) in order to obtain  $I \cap \langle \text{Itr}(I)^{\perp} \rangle_K$ . This completes the required approach for ideal representation of any finite codimension ideal in  $\mathcal{E}$ .

**Remark 3.5.** Applications of the above procedure is when  $I$  is given by a finite set of generators  $\{g_i\}_{i=1}^n$ . Thus, it is essentially useful (when it is possible) to instead use a truncated Taylor expansion. This is justified when the hypothesis of Theorem 2.16 (part 5) is satisfied. Parts (1, 3) in Theorem 2.16 indicate that for finite codimension ideals in  $\mathcal{E}$ , we may simply replace  $\mathcal{R} := \mathcal{E}$  with either  $\mathcal{R}$  or  $K[[x, \lambda]]$ . Part (6) in Theorem 2.16 provides an important computational criterion for ideals with infinite codimension in  $\mathcal{E}$ .

## 4 Computations of objects in singularity theory

In this section we recall the algebraic tools and present our suggested approaches that are needed for computation of *normal forms*, *universal unfolding*, and *persistent bifurcation diagram classification*. We acknowledge frequent fruitful discussions with Professor A. Hashemi. The second author acknowledges B. M.-Alizadeh's helps, ideas and lessons in the early stages of this project. They were essentially helpful in a fast learning of the concepts from algebraic geometry and their programming.

### 4.1 Normal form

Given a singular germ  $g \in \mathcal{E}$ , from [29, Pages 88–89] we recall the intrinsic ideals  $\mathcal{P}(g)$  and  $\mathcal{S}(g)$  by

$$\mathcal{P}(g) := \text{Itr} \left( \langle xg, \lambda g, x^2 g_x, \lambda g_x \rangle_{\mathcal{E}} \right),$$

and

$$\mathcal{S}(g) := \Sigma_{(m_i, n_i)} \mathcal{M}_{\mathcal{E}}^{m_i} \langle \lambda^{n_i} \rangle_{\mathcal{E}}, \quad (4.1)$$

where  $\frac{\partial^{m_i}}{\partial x^{m_i}} \frac{\partial^{n_i}}{\partial \lambda^{n_i}}(g)(0, 0) \neq 0$ , and there would not exist nonnegative integers  $p$  and  $q$  such that  $\frac{\partial^p}{\partial x^p} \frac{\partial^q}{\partial \lambda^q}(g)(0, 0) \neq 0$ ,  $q \leq n_i$ ,  $p + q \leq m_i + n_i$ . The extra restrictions on  $m_i$  and  $n_i$  here make the presentation (4.1) unique as of those in Equation (3.1).

[29, Proposition 8.6] indicates that for any germ  $p \in \mathcal{P}$ ,  $g \pm p$  is contact-equivalent to  $g$ . Terms in  $\mathcal{P}(g)$  are called *high order terms*; see [29, Page 89]. Therefore, we may stay contact-equivalent to  $g$  by removing all terms in  $\text{Terms}(g) \cap \mathcal{P}(g)$  from Taylor expansion of  $g$ . Therefore,  $\mathcal{P}(g)$  has a finite codimension if and only if  $g$  is *finitely determined*, i.e.,  $g$  is contact-equivalent to a polynomial germ.

[29, Theorems 8.3 and 8.4] state that for any term  $x^m \lambda^n$  in  $\mathcal{S}(g)^\perp$ , we must have  $\frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial \lambda^n}(g)(0, 0) = 0$ , while  $\frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial \lambda^n}(g)(0, 0) \neq 0$  for any intrinsic generator  $x^m \lambda^n$  in  $\mathcal{S}(g)$ . Now we recall the *intermediate order terms* as terms belonging to

$$A := \mathcal{P}(g)^\perp \setminus \left( \mathcal{S}^\perp(g) \bigcup \{x^{m_i} \lambda^{n_i} \mid x^{m_i} \lambda^{n_i} \text{ is an intrinsic generator for } \mathcal{S}(g)\} \right).$$

#### 4.1.1 Normal form computation

Now we are ready to provide an algorithm for computing *normal form* of a given germ  $g$ . Using the procedure given in Subsection 3.1, we may compute  $\mathcal{P}(g)$  and remove all terms in  $\text{Terms}(g) \cap \mathcal{P}(g)$  from Taylor expansion of  $g$  to obtain a more simplified contact-equivalent germ, say  $f$ . Now it only remains to eliminate intermediate order terms  $A$  from  $f$  as many as possible. Then, this gives rise to its *normal form*.

When  $A$  is empty,  $f$  will be called *normal form* of  $g$ . Otherwise, we may use suitable polynomial change of variable  $X(x, \lambda)$  and positive polynomial germ  $S(x, \lambda)$  to eliminate some intermediate

order terms from  $f$ . For example, we may replace  $x$  by  $ax + b\lambda$  in  $f$  where  $a > 0$  and  $b$  are arbitrary constant coefficients. This gives rise to a system of linear equations and a maximal solvable subsystem leads to the further elimination of negligible terms in  $f$  and thus, the normal form computation of  $g$ . This approach needs to be systematically adopted along with standard basis computations for high order terms in multi-state variable cases. This is because complete algebraic characterization for high order terms is not yet known for many multi-state variable cases. This will be addressed in details in our upcoming result for multi-state variable cases.

## 4.2 Universal unfolding

In this section we recall the algebraic formulation needed for computation of the universal unfolding of a singular germ  $g$ .

The tangent space  $T(g)$  is defined by

$$T(g) := \langle g, xg_x, \lambda g_\lambda \rangle_{\mathcal{E}} \oplus K\{g_x, g_\lambda, \lambda g_\lambda, \dots, \lambda^\ell g_\lambda\},$$

for sufficiently large  $\ell$ . For a finite codimension  $g$ , there exists a natural number  $\ell$  so that  $\lambda^\ell g_\lambda \notin \langle g, xg_x, \lambda g_\lambda \rangle_{\mathcal{E}}$  and  $\lambda^l g_\lambda \in \langle g, xg_x, \lambda g_\lambda \rangle_{\mathcal{E}}$  for  $l > \ell$ ; see [29, Page 127].

Next, a universal unfolding of  $g$  is defined by

$$G(x, \lambda, \mu) := g(x, \lambda) + \sum_{i=1}^k \alpha_i p_i(x, \lambda), \quad (4.2)$$

where  $\alpha := (\alpha_1, \dots, \alpha_k)$ ,  $p_i$ -s form a basis for a complement space of  $T(g)$ . Thus, we may choose  $p_i \in T(g)^\perp$ . The number  $k$  is called *codimension* of  $T(g)$  or equivalently *codimension* of  $g$ .

The tangent space  $T(g)$  is computed via Lemma 3.2 and the procedure given in Section 3.1. Since Equation (3.2) is a direct summation here, a complement space basis can be obtained through Equation (3.5) and a linear algebra computation.

## 4.3 Persistent bifurcation diagram classification

Given our description in Section 1, persistent bifurcation diagram classification is complete by simplifying the defining equations of transition set  $\Sigma$  and then, choosing one parameter from each connected component of  $\Sigma^c$ ; see [29, Page 140]. The latter is enabled in **Singularity** by using the Maple package **RegularChains**. Let  $g$  be a singular germ of finite codimension and  $G(x, \lambda, \alpha)$ ,  $\alpha \in \mathbb{R}^k$ , be a universal unfolding of the germ  $g$ . We recall that the transition set  $\Sigma := \mathcal{B} \cup \mathcal{H} \cup \mathcal{D}$ , where

$$\begin{aligned} \mathcal{B} &:= \{\alpha \in \mathbb{R}^k \mid G = G_x = G_\lambda = 0 \text{ at } (x, \lambda, \alpha) \text{ for some } (x, \lambda) \in \mathbb{R} \times \mathbb{R}\}, \\ \mathcal{H} &:= \{\alpha \in \mathbb{R}^k \mid G = G_x = G_{xx} = 0 \text{ at } (x, \lambda, \alpha) \text{ for some } (x, \lambda) \in \mathbb{R} \times \mathbb{R}\}, \\ \mathcal{D} &:= \{\alpha \in \mathbb{R}^k \mid G = G_x = 0 \text{ at } (x_i, \lambda, \alpha) \text{ for } i = 1, 2 \text{ and } x_1 \neq x_2\}. \end{aligned} \quad (4.3)$$



Possible reduction of variables in the defining equations (in particular removing  $x$  and  $\lambda$  from the equations) in (4.3) is desirable. In order to achieve this goal, we let  $I \subset K[x, \lambda, \alpha]$  and  $J \subset \mathcal{E}_{x, \lambda, \alpha}$  be the ideals generated by polynomial defining equations (4.3) for either of  $\mathcal{B}$  and  $\mathcal{H}$ . Here  $\mathcal{E}_{x, \lambda, \alpha}$  stands for all germs of smooth functions of  $x, \lambda$ , and  $\alpha$ . For the case of  $\mathcal{D}$ , a new variable  $\zeta$  is introduced and the quadratic germ  $1 - \zeta(x_1 - x_2)$  is also added to the generators of the ideal  $I$  and  $J$ . This is due to the fact that  $x_1 \neq x_2$ . Next, we compute the Gröbner basis  $G$  for  $I$  in  $K[x, \lambda, \alpha], \alpha \in \mathbb{R}^k$ , with respect to  $\alpha_j \prec_{\text{lex}} \lambda \prec_{\text{lex}} x$  for  $j = 1, \dots, k$ . Thus by [14, Theorem 2],  $G \cap K[\alpha]$  is a Gröbner basis for  $I \cap K[\alpha]$ . Hence,  $G \cap K[\alpha] \subseteq J \cap \mathcal{E}_\alpha$ . Additional restrictions on the transition sets are also obtained by the other elements of Gröbner basis  $G$ .

## 5 Main features of Singularity

The reader are referred to our user-guide [21] for a list of all functions, their capabilities, options and comprehensive information on how to work with **Singularity**. In this section we merely describe the main features of **Singularity**. The main functions (not all) are given in Tables 1 and 2. To illustrate how these functions work, let  $g(x, \lambda) := x^5 + \lambda x + \lambda^2$ . Then, **AlgObjects(g)** returns  $\mathcal{P} := \mathcal{M}^6 + \mathcal{M}^2\langle\lambda\rangle + \langle\lambda^2\rangle$ ,  $RT := \mathcal{M}^5 + \mathcal{M}\langle\lambda\rangle$ ,  $T := \mathcal{M}^5 + \mathcal{M}\langle\lambda\rangle + K\{x + 2\lambda, x^4 + \frac{1}{5}\lambda\}$ ,  $\mathcal{E}/T := K\{1, \lambda, x^2, x^3\}$ ,  $\mathcal{S} := \mathcal{M}^5 + \mathcal{M}\langle\lambda\rangle$ ,  $\mathcal{S}^\perp := K\{1, \lambda, x, x^2, x^3, x^4\}$ , and intrinsic generators of  $\mathcal{S}$  are  $x^5$  and  $x\lambda$ . For an example of computations in infinite codimensional ideals, consider restricted tangent space of  $g(x, \lambda) = \lambda^3 \sin(x)$ . **RT(g, InfCodim)** generates the restricted tangent space of  $g$  as  $\mathcal{M}\langle\lambda^3\rangle$ .

Table 1: **Singularity**'s main functions in singularity theory.

Function	Description
<b>Verify</b>	information on truncation degree and computational ring.
<b>AlgObjects</b>	$\mathcal{P}, RT, T, \mathcal{E}/T, \mathcal{S}, \mathcal{S}^\perp$ , intrinsic generators of $\mathcal{S}$ .
<b>Normalform</b>	normal forms of a given germ.
<b>UniversalUnfolding</b>	universal unfoldings of a given germ.
<b>RecognitionProblem</b>	for normal forms and universal unfoldings.
<b>TransitionSet</b>	transition set are computed and plotted or animated.
<b>PersistentDiagram</b>	plots or animates all persistent bifurcation diagrams.
<b>Transformation</b>	estimates $S$ and $X$ relating two contact-equivalent germs.
<b>Intrinsic</b>	intrinsic part of a given ideal or vector subspace of $\mathcal{E}$ .

**Verify(g)** checks a germ  $g$  for its bifurcation analysis while **Verify(G, Ideal, Vars)** checks

germ generators  $G$  of an ideal  $\langle G \rangle_{\mathcal{E}}$  for divisions or standard basis computations. In either case, it suggests suitable computational rings and a truncation degree and it verifies that their use does not lead to error according to Theorem 2.16.

**Normalform**( $g$ ) gives rise to the normal form  $x^5 + \lambda x$  while the universal unfolding

$$G_1 := x^5 + \lambda x + \alpha_1 + \alpha_2 \lambda + \alpha_3 x^2 + \alpha_4 x^3 \quad (5.1)$$

is derived by **UniversalUnfolding**( $g$ ). The command **TransitionSet**( $G_1$ ) derives the transition set associated with  $G_1$  as  $\mathcal{B} := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mid \alpha_1 = \alpha_2^5 + \alpha_2^3 \alpha_4 - \alpha_2^2 \alpha_3\}$ ,  $\mathcal{H}$  and  $\mathcal{D}$  given in Appendix 7.2.

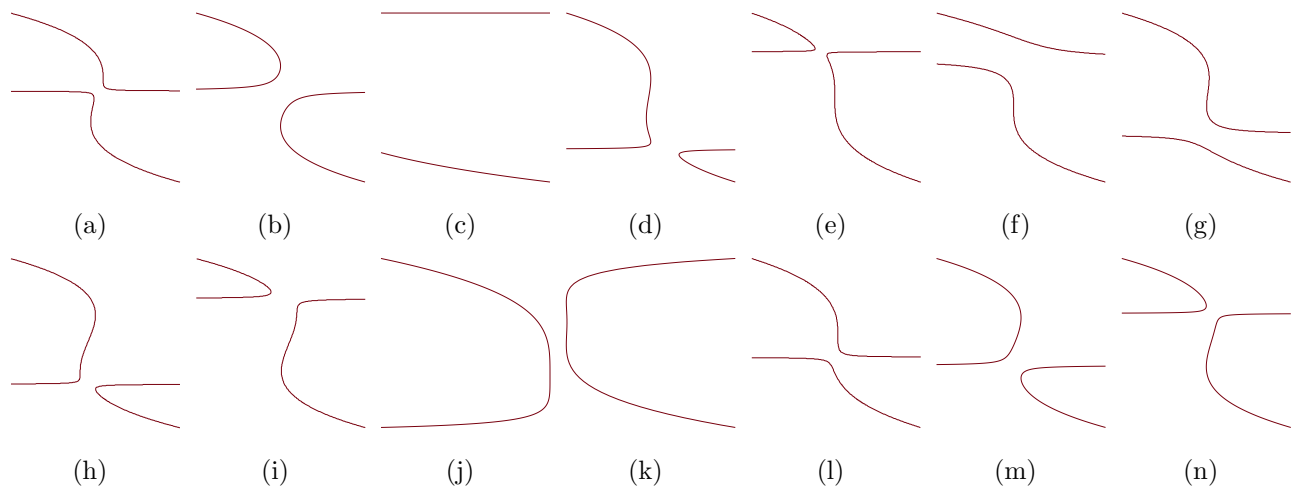


Figure 2: Persistent bifurcation diagrams for parametric singularity  $G_2$  in (5.2).

Now we consider the parametric germ

$$G_2 = x^4 + \lambda x + \alpha_1 + \alpha_2 \lambda + \alpha_3 x^2 \quad (5.2)$$

which is a universal unfolding for codimension three germ  $x^4 + \lambda x$ . **TransitionSet**( $G_2$ ) gives rise to  $\mathcal{B} := \{(\alpha_1, \alpha_2, \alpha_3) \mid \alpha_2^4 + \alpha_2^2 \alpha_3 + \alpha_1 = 0\}$ ,  $\mathcal{H} := \{(\alpha_1, \alpha_2, \alpha_3) \mid 128\alpha_2^2 \alpha_3^3 + 3\alpha_3^4 + 72\alpha_1 \alpha_3^2 + 432\alpha_1^2 = 0, \alpha_3 \leq 0\}$ , and  $\mathcal{D} := \{(\alpha_1, \alpha_2, \alpha_3) \mid \alpha_3^2 - 4\alpha_1 = 0\}$ . A list of persistent bifurcation diagrams is generated by the command **PersistentDiagram**( $G_2$ , **plot**, **ShortList**,  $[x, \lambda, \alpha_1, \alpha_2, \alpha_3]$ ); Figure 2 is some inequivalent diagrams chosen from this list.

Let

$$G := \{g_1 := \sin(\lambda^7 + x) + \exp(x^4) - x - 1 - \lambda^9, g_2 := x^5 - \lambda^2, g_3 := \cos(x^6) - \lambda - 1\},$$

**Vars** :=  $[x, \lambda]$  and  $k$  be a sufficiently large truncation degree. Then, using the commands

$$\text{StandardBasis}(G, \text{Vars}, k, \text{Fractional}), \text{StandardBasis}(G, \text{Vars}, k, \text{Formal}),$$

Table 2: Main functions of **Singularity** associated with the local rings  $K[[x, \lambda]]$ ,  $\mathcal{R}$  and  $\mathcal{E}$ .

Function	Description
<b>StandardBasis</b>	standard basis in either of the rings.
<b>ColonIdeal</b>	computes the colon ideal given in Equation (3.6).
<b>Division</b>	remainder of a germ $g$ divided by a set $G$ .
<b>MultMatrix</b>	computes the matrix $\varphi_{u,J}$ in Equations (3.4).
<b>Normalset</b>	finds a basis for complement space of an ideal $I$ .

and **StandardBasis**( $G, \text{Vars}, k, \text{SmoothGerms}$ ), we obtain the Standard basis of  $G$  in the rings of fractional maps, formal power series and smooth germs  $\mathcal{E}$ . In this example, the standard basis in either of these rings is given by  $\{\lambda, x^3\}$ . Note that the options **FormalSeries** and **Fractional** in **StandardBasis** use a truncated Taylor series expansion of germs (for non-polynomial and non-fractional germs) and are adopted according to Theorem 2.16.

## 6 Future works

Our future and in progress projects are about multi-state dimensional singularities, singularities depending on symbolic coefficients and the study of moduli, equivariant singularities with compact symmetries, and an integration of **Singularity** with the normal form analysis of singular ordinary differential equations. As our research progresses, we will update **Singularity** and its user guide [21].

In this paper, we have considered scalar singularities. However, multi-state dimensional singularities frequently occur in applications and an enhanced capability of **Singularity** for their study is an important contribution. Such enhancement enforces implementation of extended concepts and tools for module structures; see [3, 20].

The study of local zero structures for parametric smooth maps with symbolic coefficients in the vicinity of their critical points is important for applications. Here, we refer to parameters as small control parameters or perturbations while symbolic coefficients refer to those coefficients assumed to be constant. Further, the complex dynamics associated with singular germs with positive modality requires certain techniques from algebraic geometry so that they can work with symbolic coefficients and generate their associated classifications. This requires some extensions of standard basis; in our opinion, extensions similar to the notions of *comprehensive Gröbner basis* and *Gröbner system* in the context of local rings of germs; see [12, 33, 34, 36, 40]. This is in progress and will be addressed in details as a separate work.

The possible enhancement of **Singularity** for equivariant singularities with compact symme-

tries seems a challenging task but it is indeed a worthwhile contribution. We believe this should be closely related to a local germ ring version of SAGBI basis; e.g., see [37].

The contact equivalence relation is compatible with the equivalence relation arising from local invertible changes of variables and time rescaling in differential equations around an equilibrium; see [22–27, 41–43]. Thus, **Singularity** can be integrated with the normal form computations of differential equations. The aim is to locate the local bifurcation of singular differential equations in terms of the coefficients and parameters of the original systems. This contributes to the local bifurcation control and to design efficient controllers for uncontrollable engineering problems; see [26].

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## 7 APPENDIX

### 7.1 Proof of Theorem 2.15

Let  $g \in LT_{\mathcal{E}}(I) = \langle LT(f) \mid f \in I \rangle_{\mathcal{E}}$ . Since  $\mathcal{M}_{\mathcal{A}}^k \subseteq \langle p_1, \dots, p_n \rangle_{\mathcal{A}}$ , part (1) in Theorem 2.16 implies that  $\mathcal{M}_{\mathcal{E}}^k \subseteq I$  and  $\mathcal{M}_{\mathcal{E}}^k \subseteq LT_{\mathcal{E}}(I)$ . Hence, without loss of generality we may assume that  $g = J^k(g)$ . Thus,

$$g = \sum a_i LT(f_i) = J^k \sum a_i LT(f_i) = \sum J^k(a_i) LT(f_i) + r,$$

where  $a_i \in \mathcal{E}$ ,  $f_i \in I$  and  $r \in \mathcal{M}_{\mathcal{E}}^{k+1}$ . Now we only need to prove that  $LT(f_i) \in \langle LT(q_j), j = 1, \dots, m \rangle_{\mathcal{E}}$  when  $\deg(LT(f_i)) \leq k$ .

Since  $f_i \in I = \langle q_j, j = 1, \dots, m \rangle_{\mathcal{E}}$ ,  $f_i = \sum b_{ij} q_j$  for some  $b_{ij} \in \mathcal{E}$ . Thus,

$$J^k(f_i) = J^k \sum b_{ij} q_j = \sum J^k(b_{ij}) q_j + s$$

where  $s \in \mathcal{M}_{\mathcal{A}}^{k+1}$ . Therefore,  $J^k(f_i) \in \langle q_j, j = 1, \dots, m \rangle_{\mathcal{A}}$ . On the other hand,

$$\begin{aligned} LT(f_i) = LT(J^k(f_i)) &\in LT_{\mathcal{A}} \langle q_1, \dots, q_m \rangle_{\mathcal{A}} = \langle LT(q_1), \dots, LT(q_m) \rangle_{\mathcal{A}} \\ &\subseteq \langle LT(q_1), \dots, LT(q_m) \rangle_{\mathcal{E}}. \end{aligned}$$

This completes the proof.

## 7.2 Parameter spaces $\mathcal{H}$ and $\mathcal{D}$ for $G_1$ given in Equation (5.1)

$$\begin{aligned} \mathcal{H} := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mid 0 = & 3375\alpha_2^3\alpha_3^2\alpha_4^3 + 10125\alpha_1\alpha_2^2\alpha_4^4 + 16875\alpha_2^3\alpha_3^4 - 675\alpha_2^2\alpha_3^3\alpha_4^2 + 288\alpha_2\alpha_3^2\alpha_4^4 + \\ & 67500\alpha_1\alpha_2^2\alpha_3^2\alpha_4 - 900\alpha_1\alpha_2\alpha_3\alpha_4^3 + 864\alpha_1\alpha_4^5 + 1080\alpha_2\alpha_3^4\alpha_4 - 32\alpha_3^3\alpha_4^3 + 45000\alpha_1^2\alpha_2\alpha_4^2 + 13500\alpha_1\alpha_2\alpha_3^3 + \\ & 3300\alpha_1\alpha_3^2\alpha_4^2 - 108\alpha_3^5 + 30000\alpha_1^2\alpha_3\alpha_4 + 50000\alpha_1^3\} \text{ and } \mathcal{D} := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mid 2000\alpha_2^3\alpha_4^6 + 111000\alpha_2^3\alpha_3^2\alpha_4^3 - \\ & 6400\alpha_2^2\alpha_3\alpha_4^5 + 64\alpha_2\alpha_4^7 + 28000\alpha_1\alpha_2^2\alpha_4^4 - 16875\alpha_2^3\alpha_3^4 - 43200\alpha_2^2\alpha_3^3\alpha_4^2 + 5472\alpha_2\alpha_3^2\alpha_4^4 - 128\alpha_3\alpha_4^6 + \\ & 45000\alpha_1\alpha_2^2\alpha_3^2\alpha_4 - 69600\alpha_1\alpha_2\alpha_3\alpha_4^3 + 256\alpha_1\alpha_4^5 + 34020\alpha_2\alpha_3^4\alpha_4 - 1728\alpha_3^3\alpha_4^3 + 130000\alpha_1^2\alpha_2\alpha_4^2 - 81000\alpha_1\alpha_2\alpha_3^3 + \\ & 43200\alpha_1\alpha_3^2\alpha_4^2 - 5832\alpha_3^5 - 180000\alpha_1^2\alpha_3\alpha_4 + 200000\alpha_1^3 = 0\}. \end{aligned}$$